

# A Resolvent Approach to Traces and Zeta Laurent Expansions

Gerd Grubb

**ABSTRACT.** Classical pseudodifferential operators  $A$  on closed manifolds are considered. It is shown that the basic properties of the canonical trace  $\mathrm{TR} A$  introduced by Kontsevich and Vishik are easily proved by identifying it with the leading nonlocal coefficient  $C_0(A, P)$  in the trace expansion of  $A(P - \lambda)^{-N}$  (with an auxiliary elliptic operator  $P$ ), as determined in a joint work with Seeley 1995. The definition of  $\mathrm{TR} A$  is extended from the cases of noninteger order, or integer order and even-even parity on odd-dimensional manifolds, to the case of even-odd parity on even-dimensional manifolds.

For the generalized zeta function  $\zeta(A, P, s) = \mathrm{Tr}(AP^{-s})$ , extended meromorphically to  $\mathbb{C}$ ,  $C_0(A, P)$  equals the coefficient of  $s^0$  in the Laurent expansion at  $s = 0$  when  $P$  is invertible. In the mentioned parity cases,  $\zeta(A, P, s)$  is regular at all integer points. The higher Laurent coefficients  $C_j(A, P)$  at  $s = 0$  are described as leading nonlocal coefficients  $C_0(B, P)$  in trace expansions of resolvent expressions  $B(P - \lambda)^{-N}$ , with  $B$  log-polyhomogeneous as defined by Lesch (here  $-C_1(I, P) = C_0(\log P, P)$  gives the zeta-determinant).  $C_0(B, P)$  is shown to be a quasi-trace in general, a canonical trace  $\mathrm{TR} B$  in restricted cases, and the formula of Lesch for  $\mathrm{TR} B$  in terms of a finite part integral of the symbol is extended to the parity cases.

*The paper has appeared in AMS Contemp. Math. Proc. 366 (2005), 43–64. The present version includes minor corrections.*

## Introduction

The *noncommutative residue*  $\mathrm{res} A$ , the *canonical trace*  $\mathrm{TR} A$  and the *zeta-regularized determinant*  $\log \det A$  are three constants associated with the classical pseudodifferential operators ( $\psi$ do's)  $A$  on an  $n$ -dimensional closed manifold  $X$ , under various hypotheses (Wodzicki [W], Guillemin [Gu], Kontsevich and Vishik [KV], Ray and Singer [RS]). When  $P$  is an invertible elliptic classical  $\psi$ do on  $X$  of order  $m > 0$  and with spectrum in a subsector of  $\mathbb{C}$ , one can define the generalized zeta function  $\zeta(A, P, s)$  as the meromorphic extension of  $\mathrm{Tr}(AP^{-s})$  (defined for

---

2000 *Mathematics Subject Classification.* 58J42, 35S05, 58J35, 41A60.

This work was supported in part by The Danish Science Research Council, SNF grant 21-02-0446

large  $\operatorname{Re} s$ ) to  $\mathbb{C}$ ; it has a simple pole at  $s = 0$ :

$$(0.1) \quad \zeta(A, P, s) \sim \frac{1}{s} C_{-1}(A, P) + C_0(A, P) + \sum_{j \geq 1} C_j(A, P) s^j.$$

Then

$$(0.2) \quad \begin{aligned} C_{-1}(A, P) &= \frac{1}{m} \cdot \operatorname{res} A, \\ C_0(A, P) &= \operatorname{TR} A \text{ in restricted cases,} \\ C_1(I, P) &= -\log \det P. \end{aligned}$$

We shall investigate these constants, in particular  $C_0(A, P)$  and  $C_1(A, P)$ , by use of the knowledge of the structure of resolvent expressions  $A(P - \lambda)^{-N}$ . It turns out that it is rather easy to show the trace properties of  $C_0(A, P)$  using Grubb and Seeley [GS1, Th. 2.1 and 2.7], for the known cases where  $A$  is of low order, noninteger order or “odd class” (with  $n$  odd), as well as for a new case with another parity property. (Section 1.)

$\operatorname{TR} A$  does not extend to general operators of integer order  $\nu$ , but here  $C_0(A, P)$  can be viewed as a *quasi-trace*, in the sense that it is determined from  $A$  modulo local contributions (from the first  $\nu + n + 1$  homogeneous symbol terms in  $A$  and  $P$ ), and vanishes on commutators modulo local contributions. The value of  $C_0(A, P)$  modulo local terms is a finite part integral of the symbol of  $A$ , in local coordinates. (Section 2.) (The deviation of  $C_0(A, P)$  from being an independently defined trace of  $A$  is further studied in [KV], Okikiolu [O1], Melrose and Nistor [MN]; the latter call  $C_0(A, P)$  a regularized trace. See also Cardona, Ducourtieux, Magnot, Paycha [CDMP], [CDP], where it is called a weighted trace.)

In the study of  $C_1(A, P)$  and the  $C_j(A, P)$  with higher  $j$  one meets the necessity of considering resolvent expressions where  $A$  is replaced by a log-polyhomogeneous  $\psi$ do  $B$ . Here we can use the results of Lesch [L] to extend the canonical trace  $\operatorname{TR}$  to such operators  $B$ . This was done for cases with noninteger or low order in [L]; we now include also higher integer order cases with parity properties, and show that  $C_0(B, P)$  is in general a quasi-trace. In particular, we can identify  $\log \det P$  and higher derivatives of  $\zeta(I, P, s)$  at  $s = 0$  as quasi-traces; canonical traces in particular situations. (Section 3.)

Our method relies on an analysis of integrals of symbols, and involves neither comparison of meromorphic extensions nor homogeneous distributions. It moreover allows us to extend the explicit formula of Lesch [L] for the density  $\omega_{\operatorname{TR}}(B)$  defining  $\operatorname{TR} B$ , to the new integer-order cases. — The strategy is useful in situations where complex powers of operators are not easy to study directly; for example in our treatment of boundary value problems jointly with Schrohe [GSc].

## 1. The canonical trace

Consider an  $n$ -dimensional compact  $C^\infty$  manifold  $X$  without boundary. We denote  $\{0, 1, 2, \dots\} = \mathbb{N}$ . Let  $A$  be a classical (i.e., one-step polyhomogeneous)  $\psi$ do of order  $\nu \in \mathbb{R}$ , acting on the sections of a  $C^\infty$  vector bundle  $E$  over  $X$ . Let  $P$  be a classical elliptic  $\psi$ do of positive integer order  $m$ , likewise acting in  $E$  and such that the principal symbol has no eigenvalues on  $\mathbb{R}_-$ . It is shown in [GS1, Th. 2.7] by use of calculations in local coordinates that the operator family  $A(P - \lambda)^{-N}$  (for

$N > (n + \nu)/m$  has an asymptotic expansion of the trace:

$$(1.1) \quad \text{Tr}(A(P - \lambda)^{-N}) \sim \sum_{j \in \mathbb{N}} \tilde{c}_j(-\lambda)^{\frac{\nu+n-j}{m}-N} + \sum_{k \in \mathbb{N}} (\tilde{c}'_k \log(-\lambda) + \tilde{c}''_k)(-\lambda)^{-k-N},$$

for  $\lambda \rightarrow \infty$  on rays in an open subsector of  $\mathbb{C}$  containing  $\mathbb{R}_-$ .

In local coordinates, the term of degree  $\nu - mN - j$  in the symbol of  $A(P + \mu^m)^{-N}$  determines the coefficient  $\tilde{c}_j$  and, if  $\nu \in \mathbb{Z}$  and  $\frac{j-\nu-n}{m} \in \mathbb{N}$ , the coefficient  $\tilde{c}'_k$  with  $k = \frac{j-\nu-n}{m}$  (one sets the  $\tilde{c}'_k$  with  $mk + \nu + n < 0$  equal to 0). If  $\nu \notin \mathbb{Z}$ ,  $\tilde{c}'_k = 0$  for all  $k \geq 0$ . In terms of the original operators,  $\tilde{c}_j$  and (if  $\nu \in \mathbb{Z}$  and  $\frac{j-\nu-n}{m} \in \mathbb{N}$ )  $\tilde{c}'_{(j-\nu-n)/m}$  depend solely on the (strictly homogeneous part of the) homogeneous terms of degrees  $\nu, \nu - 1, \dots, \nu - j$  resp.  $m, m - 1, \dots, m - j$  in the symbols of  $A$  resp.  $P$  (in short: the first  $j + 1$  homogeneous terms). As in [GS1] and many other works, we call such coefficients “locally determined” or just “local”. The coefficients  $\tilde{c}''_k$  depend on the full structure of the operators on the manifold (are “global”).

Note that when  $\nu \in \mathbb{Z}$  and  $\frac{j-\nu-n}{m}$  is an integer  $k \geq 0$ , both  $\tilde{c}_j$  and  $\tilde{c}''_k$  contribute to the power  $(-\lambda)^{-k-N}$ . Their sum is independent of the choice of local coordinates, whereas the splitting in  $\tilde{c}_j$  and  $\tilde{c}''_k$  depends in a well-defined way on the symbol structure in the chosen local coordinates (see [GS1, Th. 2.1] or the elaboration in Theorem 1.3 below).

Along with (1.1) there is the essentially equivalent expansion (the transition between (1.1) and (1.2) is accounted for e.g. in [GS2]):

$$(1.2) \quad \Gamma(s) \text{Tr}(AP^{-s}) \sim \sum_{j \in \mathbb{N}} \frac{c_j}{s + \frac{j-\nu-n}{m}} - \frac{\text{Tr}(A\Pi_0(P))}{s} + \sum_{k \in \mathbb{N}} \left( \frac{c'_k}{(s+k)^2} + \frac{c''_k}{s+k} \right).$$

This means that  $\Gamma(s) \text{Tr}(AP^{-s})$ , defined in a standard way for  $\text{Re } s > \frac{\nu+n}{m}$ , extends meromorphically to  $\mathbb{C}$  with the pole structure indicated in the right hand side. Here  $\Pi_0$  is the orthogonal projection onto the nullspace of  $P$  (on which  $P^{-s}$  is taken to be zero). The coefficients  $\tilde{c}_j$  and  $c_j$ , resp.  $\tilde{c}'_k$  and  $c'_k$ , are proportional by universal nonzero constants. When the  $c'_k$  vanish (e.g., when  $\nu + n \notin \mathbb{N}$ ), the same holds for  $\tilde{c}'_k$  and  $c''_k$ . More generally, the pair  $\{\tilde{c}'_k, \tilde{c}''_k\}$  is for each  $k$  universally related to the pair  $\{c'_k, c''_k\}$  in a linear way. In particular,  $\tilde{c}'_0 = c'_0$ , and  $\tilde{c}''_0 = c''_0$  if  $c'_0 = 0$ , and when  $\nu$  is an integer  $\geq -n$ ,  $\tilde{c}_{\nu+n} = c_{\nu+n}$ . We shall *define*

$$(1.3) \quad \tilde{c}_{\nu+n} = c_{\nu+n} = 0 \text{ if } \nu < -n \text{ or } \nu \notin \mathbb{Z};$$

then the identifications hold in these cases too.

We are particularly interested in  $C_0(A, P)$ , defined by

$$(1.4) \quad C_0(A, P) = c_{\nu+n} + c''_0, \text{ equal to } \tilde{c}_{\nu+n} + \tilde{c}''_0 \text{ if } N = 1.$$

When  $N = 1$ ,  $C_0(A, P)$  is the coefficient of  $(-\lambda)^{-1}$  in (1.1). For general  $N$ , the coefficient of  $(-\lambda)^{-N}$  in (1.1) satisfies  $\tilde{c}_{\nu+n} + \tilde{c}''_0 = C_0(A, P) - \alpha_N c'_0$ , where  $\alpha_N = \sum_{1 \leq j < N} \frac{1}{j}$ , cf. [G5, Lemma 2.1]. [The preceding lines, modified in November 2005, correct the printed version of the present paper, where the term with  $\alpha_N$  was missing. However, in the sequel,  $C_0(A, P)$  is determined in cases where  $c'_0 = 0$ , so the results in the following remain valid.]

Division by  $\Gamma(s)$  in (1.2) gives the structure of the meromorphic extension of  $\text{Tr}(AP^{-s})$ , also denoted  $\zeta(A, P, s)$ . When  $\Pi_0(P) = 0$ , it has the Laurent expansion at  $s = 0$ :

$$(1.5) \quad \zeta(A, P, s) \sim \frac{1}{s} C_{-1}(A, P) + C_0(A, P) + \sum_{l \geq 1} C_l(A, P) s^l, \text{ with } C_{-1}(A, P) = c'_0;$$

here  $C_0(A, P)$  must be replaced by  $C_0(A, P) - \text{Tr}(A\Pi_0(P))$  if  $\Pi_0(P) \neq 0$ .

If the eigenvalues of the principal symbol of  $P$  lie in a sector  $\{\lambda \mid |\arg \lambda| \leq \theta\}$  with  $0 \leq \theta < \frac{\pi}{2}$ , so that  $e^{-tP}$  is well-defined, there is a third trace expansion that is equivalent with (1.1) and (1.2) (cf. e.g. [GS2] for the transition):

$$(1.6) \quad \text{Tr}(Ae^{-tP}) \sim \sum_{j \in \mathbb{N}} c_j t^{\frac{j-\nu-n}{m}} + \sum_{k \in \mathbb{N}} (-c'_k \log t + c''_k) t^k,$$

for  $t \rightarrow 0+$ ; the coefficients here are *the same* as those in (1.2).

One interest of studying the resolvent-type expansion (1.1) along with (1.2) is that it allows to determine the coefficients from specific integrals of symbols (cf. [GS1, Th. 2.1], Th. 1.3 below). In the following, we use the notions of [GS1] without taking space up with repetition of basic rules of calculus explained there.

In [W], Wodzicki introduced a trace functional on the full algebra of classical  $\psi$ do's, vanishing on trace-class operators; it is usually denoted  $\text{res } A$  and is called *the noncommutative residue of  $A$* . In the above situation, it satisfies

$$(1.7) \quad \text{res } A = m \cdot \text{Res}_{s=0} \text{Tr}(AP^{-s}) = m \cdot c'_0 = m \cdot \tilde{c}'_0.$$

See also Guillemin [Gu] and the survey of Kassel [K].

In [KV], Kontsevich and Vishik introduced a different trace functional  $\text{TR } A$ , called *the canonical trace*, which extends the standard trace for trace-class operators, but is only defined for part of the higher-order  $\psi$ do's. We shall show that the following definition is consistent with that of [KV]:

**DEFINITION 1.1.** *Let  $A$  be a classical  $\psi$ do in  $E$  of order  $\nu \in \mathbb{R}$ , and let  $P$  be a classical elliptic  $\psi$ do in  $E$  of even order  $m > 0$  such that the principal symbol has no eigenvalues on  $\mathbb{R}_-$ . Assume that one of the following statements is verified (with notation explained around (1.10)–(1.11) below):*

- (1)  $\nu < -n$ .
- (2)  $\nu \notin \mathbb{Z}$ .
- (3)  $\nu \in \mathbb{Z}$ ,  $A$  is even-even, and  $n$  is odd.
- (4)  $\nu \in \mathbb{Z}$ ,  $A$  is even-odd, and  $n$  is even.

*In the cases (3) and (4), take  $P$  even-even. Then*

$$(1.8) \quad \text{TR } A = \tilde{c}''_0 = c''_0 = C_0(A, P).$$

Definitions for the cases (1), (2) and (3) were given in [KV], whereas the case (4) is new. The rest of Section 1 will be devoted to the justification of Definition 1.1. [Added November 2005:] It is seen in particular that  $c'_0$  vanishes in all four cases, so that  $C_0(A, P)$  identifies directly with the coefficient of  $(-\lambda)^{-N}$  in (1.1).

In case (1), the definition is consistent with [KV] in view of the following fact:

**LEMMA 1.2.** *When  $\nu < -n$ , then  $c_{\nu+n} = c'_0 = 0$  and  $c''_0 = \text{Tr } A$ .*

PROOF. One has in this case that  $\text{Tr}(A(P - \lambda)^{-N}(-\lambda)^N) = \tilde{c}_0'' + O(\lambda^{-\varepsilon})$  with  $\varepsilon > 0$  when  $\nu < -n$ , in view of (1.1) and the comments on the coefficients. Then the identity follows since  $A(P - \lambda)^{-N}(-\lambda)^N \rightarrow A$  in trace-norm for  $\lambda \rightarrow -\infty$  on  $\mathbb{R}_-$ .

For a detailed proof of the latter fact, observe that  $R_\lambda = (P - \lambda)^{-1}$  satisfies

$$(1.9) \quad R_\lambda^N(-\lambda)^N - 1 = [(-\lambda)^N - (P - \lambda)^N](P - \lambda)^{-N} = PR_\lambda M_\lambda,$$

where  $M_\lambda = (P - \lambda)^{1-N} \sum_{0 \leq j \leq N-1} \binom{N}{j} (-\lambda)^j P^{N-1-j}$  is uniformly bounded in  $L^2(X, E)$  operator norm for  $\lambda \leq -1$ . Then with  $\delta = \min\{\frac{1}{2m}(-n - \nu), 1\} > 0$ ,

$$\begin{aligned} \|(AR_\lambda^N(-\lambda)^N - A)f\|_{H^{n+m\delta}} &\leq c\|(R_\lambda^N(-\lambda)^N - I)f\|_{H^{-m\delta}} \\ &\leq c'|\lambda|^{-\delta}|\lambda|^\delta\|R_\lambda M_\lambda f\|_{H^{m-m\delta}} \leq c''|\lambda|^{-\delta}\|f\|_{L^2}, \end{aligned}$$

using that  $|\lambda|^\delta\|R_\lambda f\|_{H^{m-m\delta}} \leq c_3(\|R_\lambda f\|_{H^m} + |\lambda|\|R_\lambda f\|_{L^2}) \leq c_4\|f\|_{L^2}$ . Thus for  $\lambda \rightarrow -\infty$ ,

$$\|AR_\lambda^N(-\lambda)^N - A\|_{\text{Tr}} \leq c_5\|AR_\lambda^N(-\lambda)^N - A\|_{\mathcal{L}(L^2, H^{n+m\delta})} \rightarrow 0. \quad \square$$

In the case (2), the formula (1.8) makes sense since  $c'_0 = 0$  and  $c_{\nu+n} = 0$  by definition. It was shown by Lesch in [L] that the definition is consistent with that of [KV] in this case, if  $P$  is selfadjoint positive with scalar leading symbol. The following analysis shows that the definition is likewise consistent with that of [KV] for the  $P$ 's considered here.

We now turn to (3) and (4): As in [G2, Sect. 5], we say that a classical  $\psi$ do  $Q$  of order  $r \in \mathbb{Z}$  with symbol  $q \sim \sum_{l \in \mathbb{N}} q_{r-l}(x, \xi)$  ( $q_{r-l} \in C^\infty$  in  $(x, \xi)$  and homogeneous of degree  $r - l$  in  $\xi$  for  $|\xi| \geq 1$ ) has **even-even** alternating parity (in short: is even-even), when the symbols with even (resp. odd) degree  $r - l$  are even (resp. odd) in  $\xi$ :

$$(1.10) \quad q_{r-l}(x, -\xi) = (-1)^{r-l} q_{r-l}(x, \xi) \text{ for } |\xi| \geq 1.$$

The operator (or symbol) is said to have **even-odd** alternating parity in the reversed situation where the symbols with even (resp. odd) degree  $r - l$  are odd (resp. even) in  $\xi$ :

$$(1.11) \quad q_{r-l}(x, -\xi) = (-1)^{r-l-1} q_{r-l}(x, \xi) \text{ for } |\xi| \geq 1.$$

Other authors use other names, e.g. [KV] calls the even-even symbols “odd-class”, studying them on odd-dimensional manifolds, and Okikiolu [O3] uses the words “regular parity” resp. “singular parity” for the even-even resp. even-odd alternating parity. Differential operators and their parametrices are even-even, whereas e.g.  $|A| = (A^2)^{\frac{1}{2}}$  is even-odd, when  $A$  is a first-order elliptic selfadjoint differential operator (as noted in [GS2, p. 48]).

In case (3), defining  $\text{TR } A$  as  $C_0(A, P)$  is consistent with [KV, Sect. 7.3], cf. also [O3]. We shall now show (from scratch) that this constant has the desired properties, and that the constant in case (4) likewise does so. The proof — inspired from [G2, Theorem 5.2] — shows that in the cases (3) and (4), the logarithmic terms and the local terms with  $\nu + n - j$  even vanish. It will be based on an exact application of the method of proof of [GS1, Th. 2.1] to the present operator family  $A(P - \lambda)^{-N}$ . Here we moreover give an account of how the coefficient  $C_0(A, P)$  looks for general  $A$ .

Write

$$(1.12) \quad A(P - \lambda)^{-N} = A(P + \mu^m)^{-N} = Q(\mu), \quad \mu = (-\lambda)^{\frac{1}{m}},$$

where  $\mu$  is included in the symbol as in [G1], [GS1], [G2]; then  $(P + \mu^m)^{-N}$  is weakly polyhomogeneous in  $(\xi, \mu)$  (strongly so if  $P$  is a differential operator).

There is a finite cover of  $X$  by coordinate patches  $U_i$  ( $i \leq i_0$ ) with trivializations of  $E$ ;  $\psi_i: E|_{U_i} \rightarrow V_i \times \mathbb{C}^{\dim E}$  with  $V_i \subset \mathbb{R}^n$ , and a subordinate partition of unity  $\varphi_j$  ( $j \leq j_0$ ) such that any two of the functions  $\varphi_j$  are supported in one of the  $U_i$ 's (for  $i = i(j_1, j_2)$ ). Then we can write

$$(1.13) \quad A = \sum_{j_1, j_2 \leq j_0} \varphi_{j_1} A \varphi_{j_2},$$

a finite sum of  $\psi$ do's, each acting in a coordinate patch (and preserving the property of being supported in the patch). Since the coefficients in the trace expansions depend linearly on  $A$ , it suffices to consider the expansions for each term in (1.13). Actually, we can, by linear translations in  $\mathbb{R}^n$ , replace the  $V_{i(j_1, j_2)}$  by sets  $V'_{j_1, j_2}$  with a positive distance from one another, so that  $A$  in (1.13) carries over to a  $\psi$ do that is a sum of pieces supported in each  $V'_{j_1, j_2}$  for  $j_1, j_2 \leq j_0$  — we shall denote it  $A$  again. We likewise consider  $(P - \lambda)^{-N}$  in the coordinate patches carried over to  $\mathbb{R}^n$  in this way.

In the localized situation, let  $Q(\mu)$  have the symbol

$$(1.14) \quad q(x, \xi, \mu) \sim \sum_{j \in \mathbb{N}} q_{\nu - mN - j}(x, \xi, \mu);$$

here the  $q_{\nu - mN - j}$  are homogeneous in  $(\xi, \mu)$  of degree  $\nu - mN - j$ , for  $|\xi| \geq 1$ . For simplicity of notation, we can let  $\mu$  run on the ray  $\mathbb{R}_+$  (other rays are treated similarly, and holomorphy in  $\mu$  is assured by [GS1, Lemma 2.3]). Besides this polyhomogeneous structure, the important knowledge is that the symbol of  $Q$  lies in a suitable  $S^{k, d}$ -space, as defined in [GS1]. When a symbol  $f(x, \xi, \mu)$  lies in  $S^{k, d}$ , the  $d$ -index indicates that  $f(x, \xi, \mu) = \mu^d f_1(x, \xi, \mu)$ , where  $f_1$  has a Taylor expansion in  $z = \frac{1}{\mu}$  at  $z = 0$ :

$$(1.15) \quad \begin{aligned} f(x, \xi, \mu) &= \mu^d f_1(x, \xi, 1/z) = z^{-d} \sum_{0 \leq l < L} f^{(l)}(x, \xi) z^l + O(\langle \xi \rangle^{k+L} z^{-d+L}) \\ &= \sum_{0 \leq l < L} f^{(l)}(x, \xi) \mu^{d-l} + O(\langle \xi \rangle^{k+L} \mu^{d-L}), \text{ for any } L, \end{aligned}$$

with  $f^{(l)} \in S^{k+l}$  (see [GS1] for further details;  $\langle \xi \rangle$  stands for  $(1 + |\xi|^2)^{\frac{1}{2}}$ ).

**THEOREM 1.3.** (i) *In the localized situation, when  $N > (\nu + n)/m$ , the diagonal value of the kernel  $K(Q(\mu), x, y)$  of  $Q(\mu)$  has an asymptotic expansion*

$$(1.16) \quad \begin{aligned} K(Q(\mu), x, x) &\sim \sum_{j \in \mathbb{N}} \tilde{c}_j(x) \mu^{\nu + n - j - mN} + \sum_{k \in \mathbb{N}} (m \tilde{c}'_k(x) \log \mu + \tilde{c}''_k(x)) \mu^{-mk - mN} \\ &\sim \sum_{j \in \mathbb{N}} \tilde{c}_j(x) (-\lambda)^{\frac{\nu + n - j}{m} - N} + \sum_{k \in \mathbb{N}} (\tilde{c}'_k(x) \log(-\lambda) + \tilde{c}''_k(x)) (-\lambda)^{-k - N}. \end{aligned}$$

Here, when we define  $\tilde{c}_{\nu+n}(x) = 0$  if  $\nu < -n$  or  $\nu \in \mathbb{R} \setminus \mathbb{Z}$ ,

$$(1.17) \quad \tilde{c}_{\nu+n}(x) + \tilde{c}_0''(x) = \oint a(x, \xi) d\xi + \text{local terms},$$

where  $\oint a$  is defined from the symbol  $a(x, \xi) \sim \sum_{j \in \mathbb{N}} a_{\nu-j}(x, \xi)$  of  $A$  by:

$$(1.18) \quad \oint a(x, \xi) d\xi = \sum_{j \leq \nu+n} \left( \int_{|\xi| \leq 1} a_{\nu-j}(x, \xi) d\xi - \frac{1-\delta_{\nu+n,j}}{\nu+n-j} \int_{|\xi|=1} a_{\nu-j}(x, \xi) dS(\xi) \right) \\ + \int_{\mathbb{R}^n} \left( a(x, \xi) - \sum_{j \leq \nu+n} a_{\nu-j}(x, \xi) \right) d\xi,$$

and the local terms depend only on the first  $n + [\nu] + 1$  strictly homogeneous terms in the symbols of  $A$  and  $P$ . When  $\nu \notin \mathbb{Z}$ , all  $\tilde{c}_k'$  vanish.

(ii) If, moreover,  $A$  and  $P$  are as in Definition 1.1 with (3) or (4), then the expansion (1.16) reduces to the form

$$(1.19) \quad K(Q(\mu), x, x) \sim \sum_{j \in \mathbb{N}, j-n-\nu \text{ odd}} \tilde{c}_j(x) (-\lambda)^{\frac{\nu+n-j}{m}-N} + \sum_{k \in \mathbb{N}} \tilde{c}_k''(x) (-\lambda)^{-k-N}.$$

(iii) In each of the cases (1)–(4) in Definition 1.1,  $\tilde{c}_0'(x) = 0$ ,  $\tilde{c}_{\nu+n}(x) = 0$  (for any choice of local coordinates), and

$$(1.20) \quad \tilde{c}_0''(x) = \oint a(x, \xi) d\xi,$$

clearly independent of  $P$ .

PROOF. In these formulas,  $\delta_{r,s}$  is the Kronecker delta, and we use the notation  $[\nu]$  for the largest integer  $\leq \nu$ . The theorem will be proved by an examination of how the coefficients in (1.1) arise in the proof of [GS1, Th. 2.1]. In the proof of (ii), we consider for definiteness e.g. the case (4) (the other case (3) is treated in a completely analogous fashion).

It is shown in [GS1, Sect. 2] that  $(P + \mu^m)^{-N}$  has symbol in  $S^{-mN,0} \cap S^{0,-mN}$ , so the symbol  $q(x, \xi, \mu)$  of  $Q(\mu) = A(P + \mu^m)^{-N}$  satisfies

$$(1.21) \quad q(x, \xi, \mu) \in S^{\nu-mN,0} \cap S^{\nu,-mN}.$$

The expansion of  $q$  corresponding to (1.15) for the  $d$ -index equal to  $-mN$  reflects the fact that

$$(1.22) \quad A(P + \mu^m)^{-N} = z^{mN} A(I + z^m P)^{-N} = z^{mN} A \sum_{0 \leq l < L} \binom{-N}{l} z^{ml} P^l + O(z^{m(N+L)}) \\ = \sum_{0 \leq l < L} \binom{-N}{l} A P^l \mu^{-m(N+l)} + O(\mu^{-m(N+L)}).$$

In fact, only  $m$ 'th powers enter nontrivially in the expansion of  $q$  (since  $q$  is a function of  $\lambda = -\mu^m$ ):

$$(1.23) \quad q(x, \xi, \mu) = \sum_{0 \leq l < L} q^{(l)}(x, \xi) \mu^{-m(N+l)} + O(\langle \xi \rangle^{\nu+mL} \mu^{-m(N+L)}).$$

Here the  $q^{(l)}$  are polyhomogeneous symbols of order  $\nu + ml$ . In case (4),  $q(x, \xi, \mu)$  is even-odd and the  $q^{(l)}$  are even-odd. (It may be observed that the first term in the last sum in (1.22) equals  $A\mu^{-mN}$ , so the first coefficient in (1.23) is  $q^{(0)} = a(x, \xi)$ .)

The kernel of  $Q(\mu) = \text{OP}(q)$ , restricted to the diagonal  $x = y$ , is

$$(1.24) \quad K_q(x, x, \mu) = \int_{\mathbb{R}^n} q(x, \xi, \mu) d\xi, \quad d\xi = (2\pi)^{-n} d\xi.$$

The contributions from a homogeneous term  $q_{\nu-mN-j}$  to (1.16) are found by splitting the corresponding integral in three pieces:

$$(1.25) \quad K_{q_{\nu-mN-j}} = \int_{|\xi| \geq \mu} q_{\nu-mN-j} d\xi + \int_{|\xi| \leq 1} q_{\nu-mN-j} d\xi + \int_{1 \leq |\xi| \leq \mu} q_{\nu-mN-j} d\xi.$$

In the first integral we replace  $\xi$  by  $\mu\eta$  and use the homogeneity; this gives a contribution to  $\tilde{c}_j(x)\mu^{\nu+n-j-mN}$ .

In case (4),  $q_{\nu-mN-j}$  is odd in  $\xi$  when  $\nu-j$  is even, so the contribution vanishes when  $\nu-j$  is even. Since  $n$  is even (as well as  $mN$ ), these are the cases where  $\nu+n-j$  is even. (In a similar way one sees in case (3) that since  $Q$  is even-even, the contributions to cases  $\nu+n-j$  even vanish since  $n$  is odd.)

For the other pieces in (1.25) we use moreover that  $q_{\nu-mN-j} \in S^{\nu-j, -mN}$ . The second piece contributes straightforwardly to the  $\tilde{c}_k''$ -terms:

$$(1.26) \quad \int_{|\xi| \leq 1} q(x, \xi, \mu) d\xi = \sum_{0 \leq l < L} \mu^{-m(N+l)} \int_{|\xi| \leq 1} q^{(l)}(x, \xi) d\xi + O(\mu^{-m(N+L)}),$$

where  $q^{(0)} = a$ , as noted above. In the third piece, the terms in the symbol are homogeneous, and since we are integrating over a bounded part of  $\mathbb{R}^n$ , we need not worry about integrability at infinity. We use the expansion (as in (1.23))

$$(1.27) \quad q_{\nu-mN-j}(x, \xi, \mu) = \sum_{0 \leq l < L} q_{\nu-j+ml}^{(l)}(x, \xi) \mu^{-mN-ml} + R_{j,L}(x, \xi, \mu),$$

where the coefficients  $q_{\nu-j+ml}^{(l)}$  are homogeneous of degree  $\nu-j+ml$  in  $\xi$  and  $R_{j,L}$  is  $O(\langle \xi \rangle^{\nu-j+ml} \mu^{-m(N+L)})$ ; in case (4), all these terms are even-odd. One finds by use of polar coordinates:

$$(1.28) \quad \begin{aligned} & \mu^{-m(N+l)} \int_{1 \leq |\xi| \leq \mu} q_{\nu-j+ml}^{(l)}(x, \xi) d\xi \\ &= \mu^{-m(N+l)} \int_1^\mu r^{\nu-j+ml+n-1} dr \int_{|\xi|=1} q_{\nu-j+ml}^{(l)}(x, \xi) dS(\xi) \\ &= \begin{cases} \frac{c_{j,l}(x)}{\nu-j+ml+n} (\mu^{\nu-j-mN+n} - \mu^{-m(N+l)}) & \text{if } \nu-j+ml+n \neq 0, \\ c_{j,l}(x) \mu^{-m(N+l)} \log \mu & \text{if } \nu-j+ml+n = 0, \end{cases} \end{aligned}$$

where

$$(1.29) \quad c_{j,l}(x) = \int_{|\xi|=1} q_{\nu-j+ml}^{(l)}(x, \xi) dS(\xi).$$

When  $j \neq \nu + ml + n$ , the term  $\frac{1}{\nu-j+ml+n} c_{j,l}(x) \mu^{\nu-j-mN+n}$  contributes to the  $\tilde{c}_j(x)$ -term, whereas  $\frac{-1}{\nu-j+ml+n} c_{j,l}(x) \mu^{-m(N+l)}$  is absorbed in the  $\tilde{c}_l''(x)$ -term. When  $j = \nu + ml + n$ , we get the  $l$ 'th log-term in the first line of (1.16) with



coefficient  $c_{j,l}(x)$ ; it will be denoted  $m\tilde{c}'_l(x)$  to comply with the notation conventions of (1.1). The second line in (1.16) is obtained by insertion of  $\mu = (-\lambda)^{\frac{1}{m}}$ . If  $\nu \notin \mathbb{Z}$ , logarithmic terms cannot occur.

Let us see how the coefficients look in case (4): The logarithmic contribution comes when  $j = \nu + ml + n$ , and then since  $n$  is even and  $q_{\nu-j+ml}^{(l)}$  is even-odd of degree  $\nu - j + ml = -n$ ,  $c_{j,l}(x) = 0$ . (Similarly, this coefficient vanishes in case (3) where  $q_{\nu-j+ml}^{(l)}$  is even-even and  $n$  is odd.) Thus there are no logarithmic terms! Moreover,  $c_{j,l}(x)$  vanishes if  $\nu - j + mN + n$  is even, i.e., when  $\nu - j + n$  is even, so there is no contribution to  $\tilde{c}_j(x)$  in this case. Hence the expansion terms from the third piece in (1.25) only contribute to the terms in (1.19).

It is accounted for in [GS1, pf. of Th. 2.1] (and in more detail in [GH, Sect. 3]) how the remainders, from the polyhomogeneous expansion (1.14) as well as the expansions in powers of  $\mu$ , are handled; for completeness we recall the arguments here: For the remainder  $R_{j,L}$  in (1.27), consider a case where  $\nu - j + mL > 0$ . As noted in [GS1],  $R_{j,L}$  is  $O(|\xi|^{\nu-j+mL}\mu^{-m(N+L)})$  for  $|\xi| \geq 1$ , so it extends by homogeneity for  $|\xi| \leq 1$  to a continuous function  $R_{j,L}^h(x, \xi, \mu)$  satisfying the same estimate. Then

$$\begin{aligned} \int_{1 \leq |\xi| \leq \mu} R_{j,L} d\xi &= \int_{|\xi| \leq \mu} R_{j,L}^h d\xi - \int_{|\xi| \leq 1} R_{j,L}^h d\xi \\ &= c'_{j,L}(x) \mu^{\nu-j-mN+n} + O(\mu^{-m(N+L)}), \end{aligned}$$

giving another contribution to the coefficient  $\tilde{c}_j(x)$ . In the case (4), the contribution vanishes for  $\nu - j + n$  even. This shows that the homogeneous terms  $q_{\nu-mN-j}$  have expansions as in (1.16) down to an  $O(\mu^{-m(N+L)})$ -error when  $L$  is large; then it holds *a fortiori* for small  $L$ . Now consider the remainder  $q'_J = q - \sum_{j < J} q_{\nu-j-mN}$  in the expansion (1.14); it is in  $S^{\nu-J-mN,0} \cap S^{\nu-J,-mN}$  (depending on  $\mu$  through  $\lambda$ ), so it has an expansion, for any  $L$ ,

$$q'_J(x, \xi, \mu) = \sum_{0 \leq l < L} q_J'^{(l)}(x, \xi) \mu^{-m(N+l)} + O(\langle \xi \rangle^{\nu-J+mL} \mu^{-m(N+L)}),$$

with  $q_J'^{(l)}(x, \xi) \in S^{\nu-J+ml}$ . Assume that  $J > \nu + mL + n$ ; then all the terms are integrable in  $\xi$ , and

$$\int_{\mathbb{R}^n} q'_J(x, \xi, \mu) d\xi = \sum_{0 \leq l < L} c'_{J,l}(x) \mu^{-m(N+l)} + O(\mu^{-m(N+L)}).$$

The  $c'_{J,l}(x)$  are taken into the coefficients  $\tilde{c}_l''(x)$ . We conclude that there is an asymptotic expansion (1.16) which for any  $L \geq 0$  can be calculated down to an error  $O(\mu^{-m(N+L)})$  by taking  $J > \nu + mL + n$ , treating the remainder  $q'_J$  as last described, and the homogeneous terms with  $j < J$  as described above.

This shows how (1.16) is obtained in general, reduced to (1.19) in case (4) (and (3)). Clearly, each  $\tilde{c}_j(x)$  depends only on the strictly homogeneous term of degree  $-mN - j$  in  $q$ , hence on the strictly homogeneous terms of the first  $j + 1$  orders in  $A$  and  $P$ .

It remains to show the formulas (1.17), (1.20) for  $\tilde{c}_0''(x)$ . Here we go back to the splitting (1.25), applied to the complete symbol  $q(x, \xi, \mu)$ , that we examine with special care. It is observed in [GS1, p. 501] that when  $P$  has symbol  $p(x, \xi)$ , the symbol  $\tilde{q}(x, \xi, \mu)$  of  $(P + \mu^m)^{-1}$  is a sum  $(p_m(x, \xi) + \mu^m)^{-1} + \tilde{q}'$ , where  $(p_m + \mu^m)^{-1} \in$

$S^{-m,0} \cap S^{0,-m}$  and  $\tilde{q}'$  has not only lower order but also better decrease in  $\mu$ :  $\tilde{q}' \in S^{-m-1,0} \cap S^{m-1,-2m}$  (since it is constructed from terms containing at least two powers of  $(p_m + \mu^m)^{-1}$ ). A similar phenomenon holds for the  $N$ 'th power of  $(P + \mu^m)^{-1}$ ; its symbol is a sum  $(p_m + \mu^m)^{-N} + \tilde{q}'_{(N)}$ , where  $(p_m + \mu^m)^{-N} \in S^{-mN,0} \cap S^{0,-mN}$  and  $\tilde{q}'_{(N)} \in S^{-mN-1,0} \cap S^{m-1,-m(N+1)}$ . The composition of  $A$  with  $\text{OP}(\tilde{q}'_{(N)})$  gives an operator with symbol in  $S^{\nu-mN-1,0} \cap S^{\nu+m-1,-m(N+1)}$  (depending on  $\mu^m$  rather than  $\mu$ ); by the preceding analysis, it has a diagonal kernel expansion of the form

$$(1.30) \quad K(x, x, \mu) \sim \sum_{j \in \mathbb{N}} d_j(x) \mu^{\nu-1+n-j-mN} + \sum_{k \geq 1} (d'_k(x) \log \mu + d''_k(x)) \mu^{-mk-mN},$$

where the sum over  $k$  starts with  $k = 1$ , so that  $\mu^{-mN}$  appears at most with a local coefficient from the series over  $j$ . In the consideration of the symbol composition

$$(1.31) \quad a(x, \xi) \circ (p_m(x, \xi) + \mu^m)^{-N} = a(x, \xi)(p_m(x, \xi) + \mu^m)^{-N} \\ + \sum_{1 \leq |\alpha| < M} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a(x, \xi) \partial_x^\alpha (p_m(x, \xi) + \mu^m)^{-N} + r_M(x, \xi, \mu),$$

we observe that whenever a derivative (in  $x$  or  $\xi$ ) hits  $(p_m + \mu^m)^{-N}$ , its  $d$ -index is lowered (since the resulting expression contains at least one more power of  $(p_m + \mu^m)^{-1}$ ) — and the same is true for the remainder  $r_M$  (constructed by Taylor expansion as in standard proofs of the composition rule). So again, the part  $\sum_{1 \leq |\alpha| < M} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a \partial_x^\alpha (p_m + \mu^m)^{-N} + r_M$  gives a kernel expansion of the form (1.30), where  $\mu^{-mN}$  appears at most in the series over  $j$ , with a local coefficient. (One could avoid this step by taking the symbol of  $(P + \mu^m)^{-N}$  in  $y$ -form, found from the conjugate transpose of the symbol of  $(P^* + \mu^m)^{-N}$ .)

It remains to consider  $\text{OP}(a(x, \xi)(p_m(x, \xi) + \mu^m)^{-N})$ . Here we remark that

$$(1.32) \quad \tilde{q}'' \equiv \mu^{-mN} - (p_m + \mu^m)^{-N} = ((p_m + \mu^m)^N - \mu^{mN})(p_m + \mu^m)^{-N} \mu^{-mN} \\ = \sum_{1 \leq l \leq N} \binom{N}{l} p_m^l \mu^{m(N-l)} (p_m + \mu^m)^{-N} \mu^{-mN};$$

a sum of terms in  $S^{m(l-N),-ml} \cap S^{ml,-m(l+N)}$ ,  $l = 1, \dots, N$ ; the sum is in  $S^{0,-m} \cap S^{mN,-m(1+N)}$ . Now write

$$(1.33) \quad a(x, \xi) = a'(x, \xi) + a''(x, \xi), \\ a' = \sum_{j \leq \nu+n} a_{\nu-j}, \quad a'' = a - \sum_{j \leq \nu+n} a_{\nu-j}.$$

Here  $a''$  is of order  $\nu - J < -n$  ( $J = \max\{n + [\nu] + 1, 0\}$ ) and defines a trace-class operator. For  $\text{OP}(a''(p_m + \mu^m)^{-N})$ , the diagonal kernel is

$$\int_{\mathbb{R}^n} a''(p_m + \mu^m)^{-N} d\xi = \mu^{-mN} \int_{\mathbb{R}^n} a'' d\xi - \int_{\mathbb{R}^n} a'' \tilde{q}'' d\xi.$$

Since  $a' \tilde{q}'' \in S^{\nu-J,-m} \cap S^{\nu-J+mN,-m(1+N)}$ , the last integral gives a series as in (1.30), now with  $\nu - 1$  replaced by  $\nu - J < -n$  so that there is no term with  $\mu^{-mN}$ . Thus the contribution from  $a''$  to the coefficient of  $\mu^{-mN}$  is  $\int a'' d\xi$ , the last parenthesis in (1.18).

For  $a'(p_m + \mu^m)^{-N}$ , we know that the integral over  $|\xi| \geq \mu$  of each  $a_{\nu-j}(p_m + \mu^m)^{-N}$  gives a local term, as in the sum over  $j$  in (1.16). For the integral over  $|\xi| \leq \mu$ , we consider the two parts  $\mu^{-mN}a'$  and  $-a'q''$  separately. The latter gives a sum of expressions

$$(1.34) \quad \binom{N}{l} \mu^{-ml} \int_{|\xi| \leq \mu} a' p_m^l (p_m + \mu^m)^{-N} d\xi,$$

where we find as in the analysis of  $q$  above that the integral alone produces terms as in (1.26) and (1.28), where nonlocal contributions start at the power  $\mu^{-mN}$ . Thanks to the extra factor  $\mu^{-ml}$  in front ( $l \geq 1$ ), (1.34) on the whole contributes locally (as in the sum over  $j$  in (1.30)) to the coefficient of  $\mu^{-mN}$ .

Finally, we study  $\mu^{-mN} \int_{|\xi| \leq \mu} a'(x, \xi) d\xi$ . The integral over  $|\xi| \leq 1$  simply gives  $\mu^{-mN} \int_{|\xi| \leq 1} a'(x, \xi) d\xi$ . The integral over  $1 \leq |\xi| \leq \mu$  of each homogeneous term is analyzed as in (1.28); here the contribution from  $a_{\nu-j}$  to the coefficient of  $\mu^{-mN}$  is  $-\frac{1}{\nu-j+n} \int_{|\xi|=1} a_{\nu-j}(x, \xi) dS(\xi)$  if  $\nu-j \neq -n$ , zero if  $\nu-j = -n$ .

Adding the contributions from  $a'$  and  $a''$  we find (1.17) with (1.18). This completes the proof of (i) and (ii).

For (iii), we just have to check the local contributions found along the way in the preceding considerations. In case (1), the series in  $j$  (as in (1.30)) begin below the power  $\mu^{-mN}$ , and in case (2), they contain only noninteger powers of  $\mu$ . In the cases (3) and (4), it is checked as in the beginning of the proof that the series in  $j$  contain only odd powers of  $\mu$ . So in all these cases, the only contributions to the coefficient of  $\mu^{-mN}$  come from  $\oint a(x, \xi) d\xi$ .  $\square$

Point (iii) in cases (1) and (2) was shown already by Lesch in [L]; he introduced the notation  $\oint a(x, \xi) d\xi$  with the following brief description: It equals the  $\mu$ -independent term  $\tilde{a}_0''(x)$  in the asymptotic expansion for  $\mu \rightarrow \infty$ :

$$(1.35) \quad \int_{|\xi| \leq \mu} a(x, \xi) d\xi \sim \sum_{j \in \mathbb{N}, j \neq \nu+n} \tilde{a}_j(x) \mu^{\nu+n-j} + \tilde{a}_0'(x) \log \mu + \tilde{a}_0''(x).$$

This description is clearly consistent with the calculation of  $\oint a d\xi$  in the above proof. Also the notation  $\text{LIM}_{\mu \rightarrow \infty} \int_{|\xi| \leq \mu} a(x, \xi) d\xi$  is used. The concept is related to Hadamard's definition of the finite part — *partie finie* — of certain integrals [H, p. 184 ff.].

Lesch moreover shows that in the cases (1) and (2), the density  $\oint a(x, \xi) d\xi |dx|$  associated with  $A$  is invariant under coordinate changes. In fact, he shows this also when  $A$  is given by an amplitude function  $a(x, y, \xi)$  (a symbol in  $(x, y)$ -form); then  $\oint a(x, x, \xi) d\xi |dx|$  is invariant. The proof of the invariance extends to the cases (3) and (4), since in the proof of [L, Lemma 5.3], the logarithmic contributions (the sum over  $l$ ) in Prop. 5.2 vanish because the terms of order  $-n$  are odd in  $\xi$ . So in all the cases,  $A$  defines a density  $\omega_{\text{TR}}(A)$  described in local coordinates by

$$(1.36) \quad \omega_{\text{TR}}(A) = \oint a(x, \xi) d\xi |dx| \text{ resp. } \omega_{\text{TR}}(A) = \oint a(x, x, \xi) d\xi |dx|,$$

when  $A$  has symbol in  $x$ -form  $a(x, \xi)$ , resp. in  $(x, y)$ -form  $a(x, y, \xi)$ .

Note that in (1.18) and (1.33), one can replace the sum over  $j \leq \nu + n$  by the sum over  $j \leq J$  for any choice of  $J \geq \nu + n$ , since

$$\oint a_{\nu-j} d\xi = \int a_{\nu-j} d\xi = \int_{|\xi| \leq 1} a_{\nu-j} d\xi + \frac{1}{j-\nu-n} \int_{|\xi|=1} a_{\nu-j} dS(\xi)$$

for  $j > \nu + n$ , by integrability and homogeneity. Note also that  $\oint a(x, \xi) d\xi = 0$  when  $a$  is polynomial in  $\xi$  (reconfirming the fact from [GS1, Th. 2.7] that the coefficient of  $(-\lambda)^{-N}$  is local when  $A$  is a differential operator).

For convenience, we recall that when  $A = I$ , the coefficients of  $(-\lambda)^{\frac{n-j}{m}-N}$  for  $j < m + n$  are simply

$$(1.37) \quad \tilde{c}_j(x) = \int_{\mathbb{R}^n} q_{-mN-j}^h(x, \xi, 1) d\xi, \quad 0 \leq j < m + n,$$

where  $q_{-mN-j}^h$  is the strictly homogeneous version of  $q_{-mN-j}$ ; a direct proof goes as in e.g. [G1, Th. 3.3.5, cf. (3.3.31), (3.3.39)] or [GS1, (2.16)], using that the  $q_{-mN-j}^h$  are integrable at  $\xi = 0$  for  $j < m + n$ . (Also noninteger  $m > 0$  are allowed here.) This includes  $\tilde{c}_n(x)$ , the coefficient of  $(-\lambda)^{-N}$ ; here  $C_0(I, P) = \int \text{tr } \tilde{c}_n(x) dx$ .

REMARK 1.4. The formulas (1.17), (1.18), (1.20) in Theorem 1.3 extend to the case where  $A$  is given by a symbol  $a(x, y, \xi)$  in  $(x, y)$ -form; then  $a(x, \xi)$  in the formulas is replaced by  $a(x, x, \xi)$ . To see this, we just need a supplement to the last part of the above proof. We split in two parts

$$(1.38) \quad a(x, y, \xi) = a'(x, y, \xi) + a''(x, y, \xi), \quad a' = \sum_{j \leq \nu+n} a_{\nu-j},$$

as in (1.33). The trace-class part is easily dealt with: As in (1.32),  $\mu^{-mN} - (P + \mu^m)^{-N}$  has symbol in  $S^{0, -(m+1)N}$ , so its composition with  $\text{OP}(a''(x, y, \xi))$  has symbol in  $S^{\nu-J, -(m+1)N}$  with  $\nu - J < -n$  (by the general composition rules for these symbol spaces), hence gives a diagonal kernel expansion as in (1.30) with  $\nu - 1$  replaced by  $\nu - J$ ; it contains no term with  $\mu^{-mN}$ . The diagonal kernel of  $\text{OP}(a''(x, y, \xi))\mu^{-mN}$  is simply  $\mu^{-mN} \int a''(x, x, \xi) d\xi$ .

For  $\text{OP}(a')$ , we observe that by the rules of calculus for  $\psi$ do's,  $\text{OP}(a'(x, y, \xi)) = \text{OP}(a_1(x, \xi)) + \text{OP}(a_2(x, y, \xi))$ , where

$$(1.39) \quad \begin{aligned} a_1(x, \xi) &= \sum_{|\alpha| < M} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha a'(x, y, \xi)|_{y=x}, \\ a_2(x, y, \xi) &= \sum_{|\alpha|=M} \frac{M(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \int_0^1 (1-t)^{M-1} \partial_y^\alpha a'(x, x+t(y-x), \xi) dt; \end{aligned}$$

we take  $M > \nu + n + 1$ . We can take the symbol  $\tilde{q}$  of  $(P + \mu^m)^{-N}$  in  $y$ -form. The first term in the first line of (1.39) is  $a'(x, x, \xi)$ , a symbol in  $x$ -form, whose effect is as described in the theorem; this gives the value in (1.17) with  $a(x, \xi)$  replaced by  $a'(x, x, \xi)$ . The other terms in the first line are also in  $x$ -form, now with a power  $\partial_\xi^\alpha$  in front. When the corresponding operators are composed with  $\text{OP}(\tilde{q}(y, \xi, \mu))$  and the kernel is calculated, we can perform an integration by parts w.r.t.  $\xi$ , placing the derivative on  $\tilde{q}$ . As noted earlier, the derivatives of  $\tilde{q}$  have symbols with  $d$ -index  $\leq -(m+1)N$ , so the resulting integrals have expansions as in (1.30), giving only local contributions to the coefficient of  $\mu^{-mN}$ . They vanish in the cases (1)–(4).

As for  $a_2$ , it is in  $(x, y)$ -form and equals a sum of  $\xi$ -derivatives  $\sum_{i=1}^n \partial_{\xi_i} a_{2,i}(x, y, \xi)$  with  $a_{2,i}$  of order  $< -n$ . Here the considerations on  $a''(x, y, \xi)$  apply, and moreover, the contributions to (1.18) vanish since  $\int_{\mathbb{R}^n} \partial_{\xi_i} a_{2,i}(x, x, \xi) d\xi = 0$  as the integral of a derivative.

**COROLLARY 1.5.** *Consider operators  $A$  and  $P$  on the manifold  $X$ , then there is an asymptotic expansion (1.1) for  $N > (n + \nu)/m$ . In the localized situation,*

$$(1.40) \quad C_0(A, P) = \tilde{c}_{\nu+n} + \tilde{c}_0'' = \int_{\mathbb{R}^n} \oint \text{tr } a(x, \xi) d\xi dx + \text{local terms};$$

*the local terms depend only on the first  $n + [\nu] + 1$  strictly homogeneous terms in the symbols of  $A$  and  $P$ . Here  $\text{tr}$  denotes fiber trace. (If the symbol of  $A$  is in  $(x, y)$ -form, the formula holds with  $a(x, x, \xi)$  instead.)*

(ii) *If, moreover,  $A$  and  $P$  are as in Definition 1.1 with (3) or (4), then the expansion (1.1) reduces to the form*

$$(1.41) \quad \text{Tr}(A(P - \lambda)^{-N}) \sim \sum_{j \in \mathbb{N}, j-n-\nu \text{ odd}} \tilde{c}_j(-\lambda)^{\frac{\nu+n-j}{m}-N} + \sum_{k \in \mathbb{N}} \tilde{c}_k''(-\lambda)^{-k-N}.$$

*In particular,  $\zeta(A, P, s)$  has no poles at integers  $s$ .*

(iii) *In each of the cases (1)–(4) in Definition 1.1,  $\tilde{c}_0' = \tilde{c}_{\nu+n} = 0$  and (cf. (1.36))*

$$(1.42) \quad C_0(A, P) = \tilde{c}_0'' = \int_X \text{tr } \omega_{\text{TR}}(A).$$

**PROOF.** Since  $A(P - \lambda)^{-N}$  is trace-class,  $\text{Tr}(A(P - \lambda)^{-N})$  can be expressed in the chosen local coordinates as the integral in  $x$  of the fiber trace of the kernel diagonal value. Then the corollary follows directly from Theorem 1.3 by integration in  $x$ . For the formula (1.42) we use the information leading to (1.36).  $\square$

Note in particular that we have obtained that  $\tilde{c}_0''$  depends only on  $A$  (not on the auxiliary operator  $P$ ) in the cases (1)–(4). See also Remark 1.8 further below.

**REMARK 1.6.** Note that in all the cases (1)–(4) in Definition 1.1,  $\zeta(A, P, s)$  is regular at  $s = 0$ , and

$$(1.43) \quad \zeta(A, P, 0) = c_0'' - \text{Tr}(A\Pi_0(P)) = \text{TR } A - \text{Tr}(A\Pi_0(P)).$$

Case (3) plays an important role in [O3]. The operators  $A$  in case (4) do not in themselves form an algebra (neither do the operators in the cases (1) and (2)), but we think that the definition is of interest anyway. As examples of case (4) we mention eta functions on an even-dimensional manifold  $X$ : Let  $D$  be a selfadjoint first-order elliptic differential operator on  $X$ , then the eta function  $\eta(D, s)$  is defined for  $\text{Re } s$  large by

$$\eta(D, s) = \text{Tr}(D|D|^{-s-1}) = \text{Tr}(D(D^2)^{-\frac{1}{2}}(D^2)^{-\frac{s}{2}}) = \zeta(D|D|^{-1}, D^2, \frac{s}{2});$$

here  $D|D|^{-1}$  (defined to vanish on the nullspace of  $D$ ) is of order 0 and even-odd. For the meromorphic extension according to (1.2), the locally determined coefficients  $c_n$  and  $c_0'$  vanish due to parity, and  $D\Pi_0(D^2) = 0$ , so

$$(1.44) \quad \eta(D, 0) = c_0'' = \text{TR}(D|D|^{-1}).$$

It will now be shown that the expression  $\text{TR } A$  defined in Definition 1.1 vanishes on commutators:

**THEOREM 1.7.** *Let  $A$  and  $A'$  be classical  $\psi$ do's of orders  $\nu$  resp.  $\nu' \in \mathbb{R}$ , and let  $P$  be a classical elliptic  $\psi$ do of even order  $m > 0$  such that the principal symbol has no eigenvalues on  $\mathbb{R}_-$ . Then*

$$(1.45) \quad \text{TR}([A, A']) = 0$$

holds in the following cases:

- (1')  $\nu + \nu' < -n$  ( $\nu + \nu' < 1 - n$  if the principal symbols commute).
- (2')  $\nu + \nu' \notin \mathbb{Z}$ .
- (3')  $\nu$  and  $\nu' \in \mathbb{Z}$ ,  $A$  and  $A'$  are both even-even or both even-odd, and  $n$  is odd.
- (4')  $\nu$  and  $\nu' \in \mathbb{Z}$ ,  $A$  is even-odd,  $A'$  is even-even, and  $n$  is even.

**PROOF.** The case (1') is an immediate consequence of the definition, since  $\text{TR}([A, A']) = \text{Tr}([A, A'])$  and the standard trace vanishes on commutators. For the other cases, we rewrite by use of suitable resolvent formulas and cyclic permutation, taking  $P$  even-even in case (3) and (4): In view of the identity

$$(1.46) \quad \partial_\lambda^k (P - \lambda)^{-1} = k! (P - \lambda)^{-k-1},$$

we have that

$$(1.47) \quad \begin{aligned} \text{Tr}([A, A'](P - \lambda)^{-N}) &= \frac{1}{(N-1)!} \text{Tr}(\partial_\lambda^{N-1} A A' (P - \lambda)^{-1} - \partial_\lambda^{N-1} A (P - \lambda)^{-1} A') \\ &= \frac{1}{(N-1)!} \text{Tr}(A \partial_\lambda^{N-1} ((P - \lambda)^{-1} [P, A'] (P - \lambda)^{-1})) \\ &= \text{Tr}\left(A \sum_{0 \leq M < N} c_{MN} (P - \lambda)^{-1-M} [P, A'] (P - \lambda)^{-N+M}\right). \end{aligned}$$

Let

$$(1.48) \quad Q(\mu) = A \sum_{0 \leq M < N} c_{MN} (P + \mu^m)^{-1-M} [P, A'] (P + \mu^m)^{-N+M}.$$

By the rules of calculus in [GS1],  $Q(\mu)$  has symbol in

$$(1.49) \quad S^{\nu+\nu'-mN, 0} \cap S^{\nu+\nu'+m, -m(N+1)}.$$

Here  $\nu + \nu'$  can be replaced by  $\nu + \nu' - 1$  if  $P$  is scalar, but this is without importance for the special trace term we are investigating. Since  $Q$  actually only depends on  $\lambda = -\mu^m$ , the symbol expansion like that of  $f'$  in (1.15) has only powers that are multiples of  $m$ . The important thing here is that the lowest  $d$ -index,  $d = -m(N+1)$ , is lower than that of  $A(P + \mu^m)^{-N}$  itself. By [GS1, Th. 2.1], there is then a trace expansion

$$(1.50) \quad \begin{aligned} \text{Tr}([A, A'](P - \lambda)^{-N}) &= \text{Tr } Q(\mu) \\ &\sim \sum_{j \in \mathbb{N}} \tilde{b}_j(-\lambda)^{\frac{\nu+\nu'+n-j}{m}-N} + \sum_{k \geq 1} (\tilde{b}'_k \log(-\lambda) + \tilde{b}''_k) (-\lambda)^{-k-N}. \end{aligned}$$

When  $\nu + \nu' \notin \mathbb{Z}$ , there is no term of the form  $c(-\lambda)^{-N}$ , so  $\text{TR}([A, A'])$  vanishes according to our definition. This takes care of the case (2').

In the cases (3') and (4'), we note that when  $A$  and  $A'$  have the same alternating parity, then  $AA'$  and  $[A, A']$  are even-even, whereas when they have opposite

parities,  $AA'$  and  $[A, A']$  are even-odd. Then we can use the information from Theorem 1.3 that the terms in the series over  $j$  vanish for  $\nu + \nu' + n - j$  even; this holds in particular for the constant term, where  $j = \nu + \nu' + n$ .  $\square$

The result of Theorem 1.7 is essentially known from [KV] in the cases (1')–(3'), but our proof is different; it will be generalized to log-polyhomogeneous operators in Section 3.

We note in passing that the proof that  $\text{TR } A$  in Definition 1.1 is independent of the choice of  $P$  could also be based on resolvent rules instead of the painstaking analysis of its value: Since  $(P - \lambda)^{-1} - (P' - \lambda)^{-1} = (P - \lambda)^{-1}(P' - P)(P' - \lambda)^{-1}$ , we can write

$$(1.51) \quad \begin{aligned} & \text{Tr}(A(P - \lambda)^{-N}) - \text{Tr}(A(P' - \lambda)^{-N}) \\ &= \frac{1}{(N-1)!} \text{Tr}(A \partial_\lambda^{N-1} ((P - \lambda)^{-1}(P' - P)(P' - \lambda)^{-1})). \end{aligned}$$

This operator family has symbol in  $S^{\nu-mN,0} \cap S^{\nu+m,-m(N+1)}$ , hence has a trace expansion as in (1.50) with  $\nu + \nu'$  replaced by  $\nu$ . From this, one can reason exactly as in the proof of Theorem 1.7.

REMARK 1.8. In the above considerations, we have kept the order of  $P$  fixed, equal to an even number. One can in fact show trace expansions with a similar structure as in (1.1), (1.2) when the order of  $P$  is an arbitrary  $m \in \mathbb{R}_+$ , cf. Loya [Lo], Grubb and Hansen [GH], which could have been taken as the point of departure. On the other hand, when  $P$  is of an arbitrary order  $m > 0$ ,  $\zeta(A, P, s) = \zeta(A, P^{m'/m}, s')$  for  $s' = sm/m'$ , so by a scaling of the complex variable  $s$  one can reduce to a situation where the order of  $P$  is a given even number, at least for positive selfadjoint operators.

## 2. A quasi-trace

The functional TR does not extend to the general case  $\nu \in \mathbb{Z}$  as a trace (cf. [KV], Lesch [L]). Yet it is still possible to make some further observations on the integer order case. Consider  $C_0(A, P)$  defined in (1.4). It coincides with  $\text{TR } A$  in the cases in Definition 1.1, but depends in general on  $P$ . However,  $C_0(A, P)$  has the independence of  $P$  and the commutator property in a weaker sense, namely:

PROPOSITION 2.1.

(i) *Let  $A$  be a classical  $\psi$ do of order  $\nu \in \mathbb{Z}$ , and let  $P$  and  $P'$  be classical elliptic  $\psi$ do of positive orders  $m$  and  $m'$  such that the principal symbols have no eigenvalues on  $\mathbb{R}_-$ . Then  $C_0(A, P) - C_0(A, P')$  is locally determined. More precisely, it depends solely on the strictly homogeneous parts of the first  $\nu + n + 1$  homogeneous terms in each of the symbols of  $A$ ,  $P$  and  $P'$ ; it vanishes if  $\nu < -n$ .*

(ii) *Let  $A$  and  $A'$  be classical  $\psi$ do's of order  $\nu$  resp.  $\nu'$  such that  $\nu + \nu' \in \mathbb{Z}$ , and let  $P$  be a classical elliptic  $\psi$ do of even order  $m > 0$  such that the principal symbol has no eigenvalues on  $\mathbb{R}_-$ . Then  $C_0([A, A'], P)$  is locally determined. More precisely it depends solely on the strictly homogeneous parts of the first  $n + \nu + \nu' + 1$  homogeneous terms in each of the symbols of  $A$ ,  $A'$  and  $P$ ; it vanishes if  $\nu + \nu' < -n$ .*

PROOF. (i). By Remark 1.8, we may assume that  $P$  and  $P'$  have the same even order  $m$ . Then the statement follows directly from Corollary 1.5 (i).

(ii). As noted in the proof of Theorem 1.7,  $\text{Tr}([A, A'](P - \lambda)^{-N})$  has an expansion (1.50). Since  $\nu + \nu'$  is integer, and the sum over  $k$  begins with  $k = 1$ ,

$$(2.1) \quad C_0([A, A'], P) = \tilde{b}_{\nu+\nu'+n},$$

which is locally determined as stated.  $\square$

Note that the expressions  $C_0(A, P) - C_0(A, P')$  and  $C_0([A, A'])$  are *pointwise* locally determined, in the sense that they can be calculated as integrals in  $x$  of locally determined functions (in local coordinates).

The proposition shows that  $C_0(A, P)$  is in general somewhat like a trace, just *modulo local contributions*. The values of  $C_0([A, A'], P)$  and  $C_0(A, P) - C_0(A, P')$  can be described in terms of certain residues, as we shall recall in Proposition 3.1 below. However, for the sake of more general situations, we believe that it has an interest to introduce a notion of quasi-trace as follows:

DEFINITION 2.2. *Let  $X$  be an  $n$ -dimensional compact manifold without boundary, provided with a  $C^\infty$  vector bundle  $E$ , let  $A$  run through an algebra of  $\psi$ do's of orders  $\nu$  on the sections of  $E$ , and let  $P$  run through an auxiliary family of elliptic  $\psi$ do's in  $E$  without principal symbol eigenvalues on  $\mathbb{R}_-$ . Consider a function  $f(A, P)$  such that for each  $P$ , it is a linear functional on the  $A$ 's. We say that  $f$  is a quasi-trace if (i) and (ii) hold:*

- (i)  *$f(A, P) - f(A, P')$  depends only on the strictly homogeneous symbols of the first  $[\nu] + n + 1$  degrees in the symbols of  $A$ ,  $P$  and  $P'$ , and vanishes for  $\nu < -n$ .*
- (ii)  *$f([A, A'], P)$  depends only on the strictly homogeneous symbols of the first  $[\nu + \nu'] + n + 1$  degrees in the symbols of  $A$ ,  $A'$  and  $P$ , and vanishes for  $\nu + \nu' < -n$ .*

By the preceding results,  $C_0(A, P)$  is a quasi-trace in this sense. (In the formulation of (i) and (ii), when  $j \leq 0$ , the set of symbols of the first  $j$  degrees is understood to be empty. Thus since  $f$  is linear in  $A$ , the statement on the vanishing is a consequence of the preceding statement.)

As shown in (i) of Theorem 1.3,  $C_0(A, P)$  moreover has a pointwise description, where it can be obtained, modulo local contributions, as an integral in  $x$  of the function  $\oint \text{tr } a(x, \xi) d\xi$ , defined from the symbol  $a(x, \xi)$  of  $A$  in the chosen local coordinates.

Scott in [Sco] uses the name  $\text{TR}_\Delta$  for a concept like  $C_0(A, P)$  with  $P = \Delta$ . We have recently been informed that the constant  $C_0(A, P)$  (and its generalizations to  $b$ -calculi) plays an important role in the manuscript of Melrose and Nistor [MN] and subsequent works, where it is called a *regularized trace*, denoted  $\widehat{\text{Tr}}(A)$  or  $\text{Tr}_P(A)$ . It is taken up in a physics context under the name of a weighted trace  $\text{tr}^P(A)$  in Cardona, Ducourtioux, Magnot, Paycha [CDMP], [CDP].

The concept can also be useful when one has a vanishing property of the relevant local contributions, as e.g. in [G3]. A generalization to manifolds with boundary is worked out in Grubb and Schrohe [GSc].



### 3. Higher Laurent coefficients of zeta functions, log-polyhomogeneous symbols

Throughout the following, we make the extra assumption that  $P$  is invertible. (This makes the statements simpler. In general, one can replace  $P$  by  $P + P_0$ , where  $P_0$  is the operator with kernel  $K(P_0, x, y) = \sum_{1 \leq j \leq \dim \ker P} \varphi'_j(x) \varphi_j^*(y)$ , where the  $\varphi'_j$  and  $\varphi_j$  denote the zero eigensections of  $P^*$  resp.  $P$ , and correct for terms stemming from  $P_0$  afterwards.) Then  $\zeta(A, P, s) = \text{Tr}(AP^{-s})$  has the Laurent expansion (1.5) at  $s = 0$ . The noncommutative residue  $\text{res}(A) = m \cdot C_{-1}(A, P)$  and the quasi-trace  $C_0(A, P)$  have been discussed above, but also the next coefficient  $C_1(A, P)$  is of particular interest; in the case  $A = I$  it equals minus the so-called zeta-determinant of  $P$ :

$$(3.1) \quad \log \det P = -\partial_s \zeta(I, P, 0) = -C_1(I, P).$$

Not only this coefficient, but the whole Laurent expansion (1.5) can be described by use of functional calculus combined with the work of Lesch [L], as we show in the following.

According to Seeley [S], the complex powers of  $P$  are defined by

$$(3.2) \quad \begin{aligned} P^{-s} &= \frac{i}{2\pi} \int_{\mathcal{C}} \lambda^{-s} (P - \lambda)^{-1} d\lambda, \text{ for } \text{Re } s > 0, \\ P^{-s} &= P^{-s-N} P^N = P^N P^{-s-N} \text{ in general;} \end{aligned}$$

here  $\mathcal{C}$  is a curve in  $\mathbb{C} \setminus \overline{\mathbb{R}}_-$  encircling the spectrum of  $P$  in the positive direction (replace intervals of  $\overline{\mathbb{R}}_-$  by small half-circles around the finitely many possible eigenvalues of  $P$  on  $\mathbb{R}_-$ ). One defines  $\log P = -\partial_s P^{-s}|_{s=0}$  (cf. e.g. [KV], [O1]); then

$$(3.3) \quad \begin{aligned} \partial_s P^{-s} &= -\log P P^{-s}, \text{ where} \\ \log P P^{-s} &= \frac{i}{2\pi} \int_{\mathcal{C}} \log \lambda \lambda^{-s} (P - \lambda)^{-1} d\lambda \text{ when } \text{Re } s > 0. \end{aligned}$$

It is known (cf. e.g. [O1]) that  $\log P$  is a  $\psi$ do such that in local coordinates,

$$(3.4) \quad \text{symbol of } \log P \sim m \log[\xi] I + \sum_{j \geq 0} b_{-j}(x, \xi)$$

with  $b_{-j}$  homogeneous in  $\xi$  of degree  $-j$ ,  $[\xi]$  smooth positive and equal to  $|\xi|$  for  $|\xi| \geq 1$ . These homogeneous terms are derived straightforwardly from the homogeneous terms in the symbol of  $(P - \lambda)^{-1}$ ; in particular,  $b_{-j}$  is determined from the first  $j + 1$  homogeneous terms in the symbol of  $P$ .

We remark, for the convenience of the reader, that one can show the following precisions of the quasi-trace property of  $C_0(A, P)$ , using  $\log P$  and the operator family  $P^{-s}$ :

PROPOSITION 3.1. *Let  $A, A', P$  and  $P'$  be as in Proposition 2.1. Then*

$$(3.5) \quad C_0(A, P) - C_0(A, P') = \frac{1}{m} \text{res}(A(-\log P + \log P')),$$

$$(3.6) \quad C_0([A, A'], P) = \frac{1}{m} \text{res}(A[\log P, A']).$$

PROOF. (3.5) is shown in [O1] and [KV]. (3.6) is shown in [MN], where its generalizations to the  $b$ -calculus play an important role; related formulas have appeared in the physics literature (see e.g. Mickelsson [M], Cederwall, Ferretti, Nilsson and Westerberg [CFNW]). The following method of proof is deduced from [MN].

We first observe the following consequence of [Gu] and [W] (in particular [Gu, Th. 7.1]): Let  $B(s)$  be a holomorphic family of classical  $\psi$ do's of order  $\alpha - s$  (for some real  $\alpha$ ), with  $B(0) = 0$ . Then the meromorphic extension of  $\text{Tr } B(s)$  (holomorphic for  $\text{Re } s > n + \alpha$ , the extension again denoted  $\text{Tr } B(s)$ ), satisfies:

$$(3.7) \quad \lim_{s \rightarrow 0} \text{Tr } B(s) = \text{res } B'(0).$$

For, setting  $C(s) = \frac{1}{s}(B(s) - B(0)) = \frac{1}{s}B(s)$ , we have that  $C(s)$  is a holomorphic family of classical  $\psi$ do's of order  $\alpha - s$  with  $C(0) = B'(0)$ . Moreover,  $Q(s) = \frac{1}{s}(C(s) - C(0)P_1^{-s})$  is another holomorphic family of classical  $\psi$ do's of order  $\alpha - s$ , when  $P_1$  is taken as an elliptic positive selfadjoint  $\psi$ do of order 1. Then

$$B(s) = sC(s) = sC(0)P_1^{-s} + s(C(s) - C(0)P_1^{-s}) = sB'(0)P_1^{-s} + s^2Q(s).$$

Now

$$\lim_{s \rightarrow 0} \text{Tr}(sB'(0)P_1^{-s}) = \text{res } B'(0),$$

by definition of the noncommutative residue, and

$$(3.8) \quad \lim_{s \rightarrow 0} \text{Tr}(s^2Q(s)) = 0,$$

by [Gu, Th. 7.1] (assuring that  $\text{Tr } Q(s)$  extends meromorphically to  $\mathbb{C}$ , with *simple* poles lying in  $\alpha - \mathbb{Z}$ ). This shows (3.7).

Applying (3.7) to the holomorphic family  $A(P^{-s/m} - (P')^{-s/m})$  of order  $\nu - s$ , we see that

$$\begin{aligned} \lim_{s \rightarrow 0} \text{Tr}(A(P^{-s} - (P')^{-s})) &= \lim_{s \rightarrow 0} \text{Tr}(A(P^{-s/m} - (P')^{-s/m})) \\ &= \frac{1}{m} \text{res}(A(-\log P + \log P')), \end{aligned}$$

since  $\partial_s P^{-s/m} = -\frac{1}{m} \log P P^{-s/m}$  (note that  $-\log P + \log P'$  is classical in view of (3.4)); this shows (3.5).

For (3.6), we apply (3.7) to the family  $A[A', P^{-s/m}]$  of order  $\nu + \nu' - s$ . Noting that for  $\text{Re } s$  large,

$$\text{Tr}([A, A']P^{-s}) = \text{Tr}(AA'P^{-s}) - \text{Tr}(AP^{-s}A') = \text{Tr}(A[A', P^{-s}])$$

by cyclic permutation, we find:

$$\lim_{s \rightarrow 0} \text{Tr}([A, A']P^{-s}) = \lim_{s \rightarrow 0} \text{Tr}(A[A', P^{-s/m}]) = \frac{1}{m} \text{res}(A[\log P, A']),$$

which shows (3.6). Here  $[\log P, A']$  is classical in view of (3.4).  $\square$

The meromorphic extension  $\text{Tr } A(s)$  for a holomorphic family  $A(s)$  of order  $\alpha - s$  coincides with  $\text{TR } A(s)$  when  $\alpha - s \notin \mathbb{Z}$ , by [KV, Th. 3.1].

[Added November 2005: A proof of Proposition 3.1 based directly on resolvent information is given in [G4].]

Consider now the higher derivatives of  $P^{-s}$ :

$$(3.9) \quad \begin{aligned} \partial_s^2 P^{-s} &= (-\log P)^2 P^{-s}, \\ &\vdots \\ \partial_s^l P^{-s} &= (-\log P)^l P^{-s}, \end{aligned}$$

which we also write as

$$(3.10) \quad \partial_s^l P^{-s} = \mathcal{P}_l(P) P^{-s}, \text{ with } \mathcal{P}_l(P) = (-\log P)^l.$$

The value of  $\partial_s^l \zeta(I, P, s)$  at  $s = 0$ , more generally the constant term of  $\partial_s^l \zeta(A, P, s)$  at  $s = 0$ , will be determined as a specific coefficient in expansions of  $\Gamma(s) \text{Tr}(A \mathcal{P}_l(P) P^{-s})$  and  $\text{Tr}(A \mathcal{P}_l(P) (P - \lambda)^{-N})$ . A framework for such calculations has been set up in Lesch [L]. With the notation for symbol spaces introduced there,  $A \mathcal{P}_l(P)$  is log-polyhomogeneous belonging to  $\text{CL}^{\nu, l}(X)$ . In local coordinates, the symbols of such operators have the structure

$$(3.11) \quad b(x, \xi) \sim \sum_{j \in \mathbb{N}} \sum_{\sigma=0}^l b_{\nu-j, \sigma}(x, \xi) \log^\sigma[\xi],$$

with  $b_{\nu-j, \sigma}$  homogeneous in  $\xi$  of degree  $\nu - j$  for  $|\xi| \geq 1$ . This defines the symbol space  $\text{CS}^{\nu, l}$ . The symbols (and the operators they define) are said to be of order  $\nu$ ; the degree of a term is the number  $\nu - j$ . Log-polyhomogeneous operators were studied earlier by Schrohe [Sc].

A generalization of [GS1, Th. 2.1] to such operators is shown in [L, Th. 3.7], which we develop further in Theorem 3.2 below. For this, we recall from [L] that the definition of the finite part integral  $\oint f(x, \xi) d\xi$  in (1.18), (1.35) can be extended to the symbols  $f(x, \xi) \in \text{CS}^{\nu, l}(\mathbb{R}^n)$ : Omit  $x$ -dependence. When  $\nu < -n$ ,  $\oint f(\xi) d\xi$  is the usual integral  $\int_{\mathbb{R}^n} f(\xi) d\xi$ ; more generally it is equal to the constant term  $p_0(0)$  in the asymptotic expansion of  $\int_{|\xi| \leq \mu} f(\xi) d\xi$  in powers and log-powers of  $\mu$ :

$$(3.12) \quad \begin{aligned} \oint f(\xi) d\xi &= p_0(0), \text{ when} \\ \int_{|\xi| \leq \mu} f(\xi) d\xi &\sim \sum_{j \in \mathbb{N}, j \neq \nu+n} p_{\nu+n-j}(\log \mu) \mu^{\nu+n-j} + p_0(\log \mu) \mu^0, \text{ for } \mu \rightarrow \infty. \end{aligned}$$

Here the  $p_{\nu+n-j}$  are polynomials of degree  $\leq l$  when  $\nu + n - j \neq 0$ , degree  $\leq l + 1$  when  $\nu + n - j = 0$ . An explicit formula for  $p_0(0)$  defined from a log-homogeneous term is worked out in [L, (5.12)]: When  $f(x, \xi) = f_\nu(x, \xi) \log^\sigma[\xi]$  with  $f_\nu$  homogeneous of degree  $\nu$  in  $\xi$  for  $|\xi| \geq 1$ , then

$$(3.13) \quad \oint f(x, \xi) d\xi = \int_{|\xi| \leq 1} f(x, \xi) d\xi + \frac{(1 - \delta_{\nu+n, 0})(-1)^{\sigma+1} \sigma!}{(\nu+n)^{\sigma+1}} \int_{|\xi|=1} f_\nu(x, \xi) dS(\xi).$$

It follows that when  $b$  is as in (3.11), then

$$\begin{aligned}
 (3.14) \quad & \oint b(x, \xi) d\xi \\
 &= \sum_{j \leq \nu+n} \sum_{0 \leq \sigma \leq l} \left( \int_{|\xi| \leq 1} b_{\nu-j, \sigma}(x, \xi) d\xi + \frac{(1-\delta_{\nu+n, j})(-1)^{\sigma+1} \sigma!}{(\nu+n-j)^{\sigma+1}} \int_{|\xi|=1} b_{\nu-j, \sigma}(x, \xi) dS(\xi) \right) \\
 &\quad + \int_{\mathbb{R}^n} \left( b(x, \xi) - \sum_{j \leq \nu+n} \sum_{0 \leq \sigma \leq l} b_{\nu-j, \sigma}(x, \xi) \log^\sigma[\xi] \right) d\xi.
 \end{aligned}$$

As in (1.18), the sum over  $j \leq \nu+n$  can be replaced by the sum over  $j \leq J$  for any choice of  $J \geq \nu+n$ .

The definition of having *even-even resp. even-odd alternating parity* is extended to symbols (3.11) to mean that

$$\begin{aligned}
 (3.15) \quad & \text{even-even: } b_{\nu-l, \sigma}(x, -\xi) = (-1)^{\nu-l} b_{\nu-l, \sigma}(x, \xi) \text{ for } |\xi| \geq 1, \text{ resp.} \\
 & \text{even-odd: } b_{\nu-l, \sigma}(x, -\xi) = (-1)^{\nu-l-1} b_{\nu-l, \sigma}(x, \xi) \text{ for } |\xi| \geq 1,
 \end{aligned}$$

with similar properties of the derivatives. Then we can consider the four cases in Definition 1.1 for log-polyhomogeneous operators  $B$ .

**THEOREM 3.2.** (i) *Let  $\nu \in \mathbb{R}$  and  $l \in \mathbb{N}$ , let  $B$  be log-polyhomogeneous in  $\text{CL}^{\nu, l}$  with symbol (3.11) on  $\mathbb{R}^n$ . Let  $P$  be a classical  $\psi$ do, uniformly elliptic of integer order  $m > 0$  on  $\mathbb{R}^n$  and with no principal symbol eigenvalues in a sector around  $\mathbb{R}_-$ . Let  $N > (\nu+n)/m$ . There is an asymptotic expansion of the kernel of  $B(P-\lambda)^{-N}$  at  $x=y$ :*

$$(3.16) \quad K(B(P-\lambda)^{-N}, x, x) \sim \sum_{j=0}^{\infty} \sum_{\sigma=0}^{l+1} \tilde{c}_{j, \sigma}(x) (-\lambda)^{\frac{\nu+n-j}{m}-N} \log^\sigma(-\lambda) + \sum_{k=0}^{\infty} \tilde{c}_k''(x) (-\lambda)^{-k-N};$$

here  $\tilde{c}_{j, l+1}(x) = 0$  unless  $\frac{j-\nu-n}{m} \in \mathbb{N}$ . The coefficients  $\tilde{c}_{j, \sigma}(x)$  depend on the homogeneous or log-homogeneous symbols of the first  $j+1$  degrees in  $B$  and  $P$  (are local in this sense). The  $\tilde{c}_k''$  depend on the full structure (are global). In particular, when we define  $\tilde{c}_{\nu+n, 0}(x) = 0$  if  $\nu < -n$  or  $\nu \in \mathbb{R} \setminus \mathbb{Z}$ ,

$$(3.17) \quad \tilde{c}_{\nu+n, 0}(x) + \tilde{c}_0''(x) = \oint b(x, \xi) d\xi + \text{local terms.}$$

(ii) *It follows that in the comparison of the coefficients for two choices of auxiliary operator  $P$  and  $P'$ ,  $\tilde{c}_0''(B, P, x) - \tilde{c}_0''(B, P', x)$  is local (in the above sense).*

(iii) *If, moreover,  $B$  and  $P$  satisfy (3) or (4) of Definition 1.1 (in particular,  $m$  is even), then the expansion (3.16) reduces to the form*

$$\begin{aligned}
 (3.18) \quad & K(B(P-\lambda)^{-N}, x, x) \\
 & \sim \sum_{j \in \mathbb{N}, j-\nu-n \text{ odd}} \sum_{\sigma=0}^l \tilde{c}_{j, \sigma}(x) (-\lambda)^{\frac{\nu+n-j}{m}-N} \log^\sigma(-\lambda) + \sum_{k=0}^{\infty} \tilde{c}_k''(x) (-\lambda)^{-k-N};
 \end{aligned}$$

(iv) *In each of the cases (1)–(4) of Definition 1.1, the  $\tilde{c}_{\nu+n, \sigma}(x)$  vanish and*

$$(3.19) \quad \tilde{c}_0''(x) = \oint b(x, \xi) d\xi.$$

(v) If the symbol of  $B$  is given in  $(x, y)$ -form  $b(x, y, \xi)$ , the formulas hold with  $b(x, \xi)$  replaced by  $b(x, x, \xi)$ .

PROOF. The proof of [L, Th. 3.7] is in fact modeled very closely after the proof of [GS1, Th. 2.1], which we recalled to a large extent in the proof of Theorem 1.3 above. Let

$$(3.20) \quad Q(\mu) = B(P - \lambda)^{-N}, \text{ with } -\lambda = \mu^m.$$

The symbol  $q(x, \xi, \mu)$  is now the composite of a log-polyhomogeneous symbol in  $\text{CS}^{\nu, l}$  and a weakly polyhomogeneous symbol in  $S^{-mN, 0} \cap S^{0, -mN}$ . Here, since  $\text{CS}^{\nu, l} \subset S^{\nu+\varepsilon}$  for any  $\varepsilon > 0$ ,

$$(3.21) \quad q(x, \xi, \mu) \in S^{\nu+\varepsilon-mN, 0} \cap S^{\nu+\varepsilon, -mN}$$

(defined also for  $S_{1,0}$ -symbols without requirements of polyhomogeneity). Now there are expansions

$$(3.22) \quad q(x, \xi, \mu) = \mu^{-mN} \sum_{0 \leq l' < L} q^{(l')}(x, \xi) \mu^{-ml'} + O(\langle \xi \rangle^{\nu+\varepsilon+mL} \mu^{-m(N+L)}),$$

for all  $L$ , with coefficient symbols  $q^{(l')}(x, \xi)$  log-polyhomogeneous in  $\text{CS}^{\nu+ml', l}$ . One analyzes the kernel defined from  $q$  by (1.24), by splitting the contribution from each log-homogeneous term into three pieces as in (1.25). The integral over  $\{|\xi| \geq \mu\}$  contributes to the  $\tilde{c}_{j,\sigma}$ -terms. For the integral over  $\{|\xi| \leq 1\}$  one uses (3.22) and gets contributions to the  $\tilde{c}''_k$ -terms. For the integral over  $\{1 \leq |\xi| \leq \mu\}$  one likewise uses the expansion (3.22) for each log-homogeneous term in the symbol; each expansion term gives a contribution

$$(3.23) \quad \begin{aligned} & \mu^{-m(N+l')} \int_{1 \leq |\xi| \leq \mu} q_{\nu-j+ml', \sigma}^{(l')}(x, \xi) \log^\sigma |\xi| d\xi \\ &= \mu^{-m(N+l')} \int_1^\mu r^{\nu-j+ml'+n-1} \log^\sigma r dr \int_{|\xi|=1} q_{\nu-j+ml', \sigma}^{(l')}(x, \xi) dS(\xi) \\ &= \begin{cases} \sum_{\sigma'=0}^\sigma c_{j,l',\sigma'} \mu^{\nu-j-mN+n} \log^{\sigma'} \mu + c_{j,l'}'' \mu^{-m(N+l')} & \text{if } \nu-j+ml'+n \neq 0, \\ c_{j,l'}' \mu^{-m(N+l')} \log^{\sigma+1} \mu & \text{if } \nu-j+ml'+n = 0. \end{cases} \end{aligned}$$

The coefficients  $c_{j,l',\sigma'}$  and  $c_{j,l'}'$  contribute to the  $\tilde{c}_{j,\cdot}$ -terms, whereas the coefficient  $c_{j,l'}''$  contributes to the  $\tilde{c}''_{l'}$ -term. They are proportional to  $\int_{|\xi|=1} q_{\nu-j+ml', \sigma}^{(l')}(x, \xi) dS$  by universal factors; the value of  $c_{j,l'}''$  is

$$(3.24) \quad c_{j,l'}''(x) = \frac{(-1)^{\sigma+1} \sigma!}{(\nu-j+ml'+n)^{\sigma+1}} \int_{|\xi|=1} q_{\nu-j+ml', \sigma}^{(l')}(x, \xi) dS(\xi),$$

cf. [L, (5.12)]. (This term was left out in [L, (3.38)]; the connection between  $\tilde{c}''_0$  and  $\tilde{f}b$  was made only towards the end of the paper.) Remainders are treated essentially as in [GS1], as recalled above in Theorem 1.3; this shows (3.16).

In the cases (3) and (4) as in Definition 1.1, the terms in the sum over  $j$  vanish for  $j-n-\nu$  even, since they are obtained by integration in  $\xi$  of odd functions (like in Theorem 1.3); this shows (iii).

The analysis leading to (3.17) is practically the same as in the proof of (1.17) in Theorem 1.3, only with  $a$  replaced by  $b$ ;  $P$  is unchanged. It is seen again that

all parts of  $(P + \mu^m)^{-N}$  except  $\text{OP}((p_m(y, \mu) + \mu^m)^{-N})$  gives series with a locally determined constant term, (1.30) being replaced by

$$(3.25) \quad K(x, x, \mu) \sim \sum_{j=0}^{\infty} \sum_{\sigma=0}^{l+1} \tilde{d}_{j,\sigma}(x)(-\lambda)^{\frac{\nu+n-j}{m}-N} \log^{\sigma}(-\lambda) + \sum_{k=1}^{\infty} \tilde{d}_k''(x)(-\lambda)^{-k-N}.$$

The considerations on the integral  $\int_{|\xi| \leq \mu} b(x, \xi)(p_m(x, \xi) + \mu^m)^{-N} d\xi$  carry over verbatim from the considerations on  $\int_{|\xi| \leq \mu} a(x, \xi)(p_m(x, \xi) + \mu^m)^{-N} d\xi$  in Theorem 1.3.

This shows (3.17), and (ii) is an immediate consequence since the  $\tilde{c}_{\nu+n,0}$  are local and the symbol integrals cancel out. Moreover, (iv) is seen by observing that the local contributions (from integrals over  $|\xi| \geq \mu$  and from the sum over  $j$  in the various series of the form (3.25) that arise in the analysis) vanish in the cases (1)–(4) of Definition 1.1.

Finally, (v) is included as in Remark 1.4.  $\square$

It is known from [L] in the cases (1) and (2) for log-polyhomogeneous operators that the density  $\oint b(x, \xi) d\xi |dx|$  or  $\oint b(x, x, \xi) d\xi |dx|$  has an invariant meaning, and the argument carries over to log-polyhomogeneous operators in the parity cases (3) and (4), in the same way as mentioned after Theorem 1.3. So, in the cases (1)–(4), when  $B$  is given on  $X$ , it defines a density  $\omega_{\text{TR}}(B)$  described in local coordinates by

$$(3.26) \quad \omega_{\text{TR}}(B) = \oint B(x, \xi) d\xi |dx| \quad \text{resp.} \quad \omega_{\text{TR}}(B) = \oint b(x, x, \xi) d\xi |dx|,$$

when  $B$  has symbol in  $x$ -form  $b(x, \xi)$ , resp. in  $(x, y)$ -form  $b(x, y, \xi)$ .

The inclusion of symbols in  $(x, y)$ -form allows us in particular to observe that when  $B_1 B_2$  is as in one of the cases (1)–(4), with  $B_1 = \text{OP}(b_1(x, \xi))$  and  $B_2 = \text{OP}(b_2(y, \xi))$  in a local coordinate system, then

$$(3.27) \quad \omega_{\text{TR}}(B_1 B_2) = \oint b_1(x, \xi) b_2(x, \xi) d\xi |dx|.$$

We have as usual a corollary on the manifold situation, when  $B$  is decomposed as in (1.13) and the pieces are carried over to local coordinates in  $\mathbb{R}^n$  as explained there.

**COROLLARY 3.3.** *Consider a log-polyhomogeneous operator  $B$  on the manifold  $X$ , together with  $P$  as in Proposition 2.1, with  $N > (\nu + n)/m$ . Then there is an asymptotic expansion of the trace:*

$$(3.28) \quad \text{Tr}(B(P - \lambda)^{-N}) \sim \sum_{j \in \mathbb{N}} \sum_{\sigma=0}^{l+1} \tilde{c}_{j,\sigma}(-\lambda)^{\frac{\nu+n-j}{m}-N} \log^{\sigma}(-\lambda) + \sum_{k=0}^{\infty} \tilde{c}_k''(-\lambda)^{-k-N}.$$

Here, when we define  $\tilde{c}_{\nu+n,0} = 0$  if  $\nu < -n$  or  $\nu \in \mathbb{R} \setminus \mathbb{Z}$ , and consider the operators localized to  $\mathbb{R}^n$  as explained before Theorem 1.3, then

$$(3.29) \quad \tilde{c}_{\nu+n,0} + \tilde{c}_0'' = \int_{\mathbb{R}^n} \oint \text{tr } b(x, \xi) d\xi dx + \text{local terms};$$

where  $b(x, \xi)$  is the symbol of  $B$ ; the local terms depend only on the strictly homogeneous terms in the symbols of  $B$  and  $P$  for  $j \leq n + [\nu]$ . (If the symbol  $b$  is in  $(x, y)$ -form, the formula holds with  $b(x, x, \xi)$  instead.)

(ii) It follows that in the comparison of the coefficients for two choices of auxiliary operator  $P$  and  $P'$ ,  $\tilde{c}_{\nu+n,0}(B, P) + \tilde{c}_0''(B, P) - (\tilde{c}_{\nu+n,0}(B, P') + \tilde{c}_0''(B, P'))$  is local.

(iii) If, moreover,  $B$  and  $P$  satisfy (3) or (4) of Definition 1.1, then the expansion reduces to the form

$$(3.30) \quad \text{Tr}(B(P-\lambda)^{-N}) \sim \sum_{j \in \mathbb{N}, j-n-\nu \text{ odd}} \sum_{\sigma=0}^l \tilde{c}_{j,\sigma}(-\lambda)^{\frac{\nu+n-j}{m}-N} \log^\sigma(-\lambda) + \sum_{k=0}^{\infty} \tilde{c}_k''(-\lambda)^{-k-N}.$$

In particular,  $\zeta(B, P, s)$  (cf. (3.32) below) is regular at all integers  $s$ .

(iv) In each of the cases (1)–(4) in Definition 1.1, the  $\tilde{c}_{\nu+n,\sigma}$  vanish (for any choice of local coordinates), and (cf. (3.26))

$$(3.31) \quad \tilde{c}_0'' = \int_X \text{tr } \omega_{\text{TR}}(B).$$

In the following, we draw on the hypothesis that  $P$  is invertible. By the transition formulas in [GS2], (3.28) implies the structure of the meromorphic extension of  $\Gamma(s) \text{Tr}(BP^{-s})$ , also denoted  $\Gamma(s)\zeta(B, P, s)$ :

$$(3.32) \quad \Gamma(s) \text{Tr}(BP^{-s}) = \Gamma(s)\zeta(B, P, s) \sim \sum_{j=0}^{\infty} \sum_{\sigma=0}^{l+1} \frac{c_{j,\sigma}}{(s + \frac{j-\nu-n}{m})^{\sigma+1}} + \sum_{k=0}^{\infty} \frac{c_k''}{s+k},$$

with universal nonzero factors linking  $\tilde{c}_{j,\sigma}$  with  $c_{j,\sigma}$  and  $\tilde{c}_k''$  with  $c_k''$ ; in particular,  $c_0'' = \tilde{c}_0''$  and  $c_{\nu+n,0} = \tilde{c}_{\nu+n,0}$  (with the usual zero convention if the series in  $j$  does not contain such a term). Dividing out  $\Gamma(s)$ , we see that  $\zeta(B, P, s)$  has a Laurent expansion at  $s = 0$  when  $B \in \text{CL}^{\nu,l}(X)$ :

$$(3.33) \quad \zeta(B, P, s) \sim \sum_{r \geq -l-1} C_r(B, P) s^r;$$

here

$$(3.34) \quad C_0(B, P) = c_{\nu+n,0} + c_0'' = \tilde{c}_{\nu+n,0} + \tilde{c}_0''.$$

Corollary 3.3 (ii) shows that  $C_0(B, P)$  satisfies the first condition in Definition 2.2 for being a quasi-trace. (In [O1], the difference  $C_0(B, P) - C_0(B, P')$  is shown to be a certain residue, in the case where the symbol of  $B$  equals  $c \log[\xi]$  plus a zero-order classical symbol.) We shall now show that the second condition, concerning commutators, is likewise satisfied.

**THEOREM 3.4.** *Let  $B \in \text{CL}^{\nu,l}$ ,  $B' \in \text{CL}^{\nu',l'}$ , and let  $P$  be as in Proposition 2.1 (ii). Then*

$$(3.35) \quad \text{Tr}([B, B'](P-\lambda)^{-N}) \sim \sum_{j=0}^{\infty} \sum_{\sigma=0}^{l+1} \tilde{b}_{j,\sigma}(-\lambda)^{\frac{\nu+\nu'+n-j}{m}-N} \log^\sigma(-\lambda) + \sum_{k=1}^{\infty} \tilde{b}_k''(-\lambda)^{-k-N}.$$

Hence the zeta function  $\zeta([B, B'], P, s) = \text{Tr}([B, B']P^{-s})$  satisfies (when  $P$  is invertible)

$$(3.36) \quad \zeta([B, B'], P, s) \sim \sum_{r \geq -l+l'-1} C_r([B, B'], P) s^r \text{ for } s \rightarrow 0,$$

with  $C_0([B, B'], P)$  local, depending solely on the terms of the first  $n + [\nu + \nu'] + 1$  homogeneity degrees in the symbols of  $B$ ,  $B'$  and  $P$ .

In particular, it vanishes if one of the conditions (1')–(4') in Theorem 1.7 is satisfied (with  $A$ ,  $A'$  replaced by  $B$ ,  $B'$ ).

PROOF. Since  $[B, B']$  has symbol in  $\text{CS}^{\nu+\nu', l+l'}$ , we know already from Corollary 3.3 that  $Q = [B, B'](P - \lambda)^{-N}$  has the corresponding expansion (3.28); we just have to show that  $\tilde{c}_0'' = 0$  there.

As in (1.47) and (1.48), the trace calculation can be reduced to the calculation for

$$(3.37) \quad Q(\mu) = B \sum_{0 \leq M < N} c_{MN}(P + \mu^m)^{-1-M} [B, B'](P + \mu^m)^{-N+M};$$

by the rules of calculus, it has symbol in  $S^{\nu+\nu'-mN+\varepsilon, 0} \cap S^{\nu+\nu'+m+\varepsilon, -(m+1)N}$ . The trace expansion can be analyzed as in the proof of Theorem 3.2; with the modification that the contributions from the integral over  $\{|\xi| \leq 1\}$  start with the lower power  $\mu^{-m(N+1)}$ , and the terms of the form  $-c_{j,l}' \mu^{-\alpha}$  from (3.23) all have  $\alpha \geq m(N+1)$ . Thus  $\tilde{c}_0'' = 0$  for this operator family. In the special cases (1')–(4'), also the contribution from the series in  $j$  vanishes, as shown in Corollary 3.3.  $\square$

We have hereby obtained:

COROLLARY 3.5. (i) For log-polyhomogeneous operators  $B$  with  $P$  as in Proposition 2.1,  $C_0(B, P)$  (3.34) is a quasi-trace in the sense of Definition 2.3.

(ii) Definition 1.1 of the canonical trace  $\text{TR}$  extends to log-polyhomogeneous operators  $B$ , in such a way that the definition is independent of  $P$ , Theorem 1.7 extends to these operators, and

$$(3.38) \quad \text{TR}(B) = \int_X \text{tr } \omega_{\text{TR}}(B).$$

Point (ii) was shown in [L, Sect. 5] for the cases where  $\nu \notin \mathbb{Z}$  or  $< -n$ , by a somewhat different proof (where  $P$  was assumed to be selfadjoint positive with scalar principal symbol).

Let us now return to the zeta function for a classical  $\psi$ do, cf. (3.1). The expansions (3.28) and (3.32) hold in particular for  $B = A\mathcal{P}_l(P)$ ; let us denote the coefficients in this case by  $\tilde{c}_{j,\sigma}^{(l)}$ ,  $\tilde{c}_k^{\prime(l)}$ , resp.  $c_{j,\sigma}^{(l)}$ ,  $c_k^{\prime(l)}$ . Then we have found that

$$(3.39) \quad \partial_s^l \zeta(A, P, s) = \text{Tr}(A\mathcal{P}_l(P)P^{-s}) = \zeta(A\mathcal{P}_l(P), P, s)$$

has the meromorphic structure determined from

$$(3.40) \quad \Gamma(s) \text{Tr}(A\mathcal{P}_l(P)P^{-s}) \sim \sum_{j=0}^{\infty} \sum_{\sigma=0}^{l+1} \frac{c_{j,\sigma}^{(l)}}{(s + \frac{j-\nu-n}{m})^{\sigma+1}} + \sum_{k=0}^{\infty} \frac{c_k^{\prime(l)}}{s+k}.$$



Concerning  $\zeta(A, P, s)$  in (3.1), we find by differentiation in  $s$ :

$$(3.41) \quad \zeta(A\mathcal{P}_l(P), P, s) = \partial_s^l \zeta(A, P, s) \sim \frac{(-1)^l l!}{s^{l+1}} C_{-1}(A, P) + \sum_{r \geq l} \frac{r!}{(r-l)!} C_r(A, P) s^{r-l},$$

so that  $C_0(A\mathcal{P}_l(P), P) = l! C_l(A, P)$  for  $l \geq 1$ . Dividing by  $\Gamma(s)$  in (3.40), we see that the constant term at  $s = 0$  in the expansion of  $\zeta(A\mathcal{P}_l(P), P, s)$  is  $c_{\nu+n,0}^{(l)} + c_0''^{(l)}$ , with the usual conventions. So we have found:

COROLLARY 3.6. *For  $l \geq 1$ ,*

$$(3.42) \quad C_l(A, P) = \frac{1}{l!} C_0(A\mathcal{P}_l(P), P) = \frac{1}{l!} (c_{\nu+n,0}^{(l)} + c_0''^{(l)}),$$

*defined from the constants appearing in (3.40); here  $c_{\nu+n,0}^{(l)} = 0$  if  $\nu + n \notin \mathbb{N}$  or  $\nu < -n$ .*

Note that  $c_{\nu+n,0}^{(l)}$  is locally determined (in local coordinates it comes from the part of the symbol of  $A\mathcal{P}_l(P)(P + \mu^m)^{-N}$  with homogeneity degree  $-mN - n$ ), whereas  $c_0''^{(l)}$  depends on the full structure (is global).

It may also be observed that the coefficient of  $s^{-l-1}$  in (3.41), proportional to  $\text{res } A$ , is also proportional to the  $(l+1)$ 'st higher residue of  $A\mathcal{P}_l(P)$  defined in [L, Sect. 4].

In particular,  $C_1(A, P)$  equals  $c_{\nu+n,0}^{(1)} + c_0''^{(1)}$ ; this specializes to a formula for  $-\log \det P$  in the case  $A = I$ :

COROLLARY 3.7. *One has that*

$$(3.43) \quad -\log \det P = C_1(I, P) = C_0(-\log P, P) = c_{\nu+n,0}^{(1)} + c_0''^{(1)},$$

*determined from (3.40) in the case  $A = I$ .*

One cannot conclude from Corollary 3.6 that the higher Laurent coefficients in (3.1) are quasi-traces of  $A$  itself — for in  $C_0(A\mathcal{P}_l(P), P)$ , the first entry depends highly on the choice of  $P$  when  $l > 0$ . However, when parity and dimension match, these coefficients can be expressed by the extended TR applied to  $A\mathcal{P}_l(P)$ . In fact, when  $P$  is even-even,  $\mathcal{P}_l(P)$  is even-even for all  $l \geq 1$ . Then one gets, by Corollary 3.6:

COROLLARY 3.8. *Let  $P$  be even-even. If  $n$  is odd, let the classical  $\psi$ do  $A$  be even-even; if  $n$  is even, let  $A$  be even-odd. Then*

$$(3.44) \quad C_l(A, P) = \frac{1}{l!} C_0(A\mathcal{P}_l(P), P) = \frac{1}{l!} \text{TR}(A\mathcal{P}_l(P)), \text{ for } l \geq 1.$$

*In particular:*

(i) *When  $n$  is odd, then*

$$(3.45) \quad \begin{aligned} \log \det P &= \text{TR}(\log P), \\ \partial_s^l \zeta(I, P, 0) &= \text{TR}(\mathcal{P}_l(P)), \text{ for } l \geq 1. \end{aligned}$$

(ii) *When  $n$  is even and  $D$  is a first-order selfadjoint elliptic differential operator (cf. Remark 1.4),*

$$(3.46) \quad \partial_s^l \eta(D, 0) = 2^{-l} \text{TR}(D|D|^{-1} \mathcal{P}_l(D^2)), \text{ for } l \geq 1.$$

A formula similar to the first line in (3.45) appears in the abstract of [O3] (it can be justified on the basis of [O2], Lemma 0.1). As an example of the second line,

$$(3.47) \quad \partial_s^2 \zeta(I, P, 0) = \text{TR}((\log P)^2).$$

cf. (3.10). See also [KV, Sect. 4].

REMARK 3.9. Let us set the above methods in relation to the results of [O1], [O2] and [KV] on the multiplicative anomaly  $\det AB / (\det A \det B)$  of the determinant. [O1] shows that when the elliptic positive-order  $\psi$ do's  $A$ ,  $B$  and  $AB$  have scalar principal symbol taking no values on  $\mathbb{R}_-$ , and have no eigenvalues on  $\overline{\mathbb{R}}_-$ , then

$$(3.48) \quad \log AB - \log A - \log B = [A, C^{(I)}(A, B)] + [B, C^{(II)}(A, B)] + F,$$

where  $C^{(I)}(A, B)$  and  $C^{(II)}(A, B)$  are Lie polynomials in  $A$  and  $B$ , and  $F$  is an operator of order  $< -n$  with  $\text{Tr } F = 0$ . In [O2] this is used to show that  $\log \det AB - \log \det A - \log \det B$  equals the noncommutative residue of a certain operator derived from  $A$  and  $B$ ; in particular it is locally determined. By variational methods, [O2] and [KV] show local determinedness of  $\log \det AB - \log \det A - \log \det B$  also in cases where the principal symbol is not scalar. This implies local determinedness of  $\det AB / (\det A \det B)$  (by exponentiation).

From our point of view, (3.48) implies that  $C_0(\log AB - \log A - \log B, P)$  is locally determined for any auxiliary operator  $P$ , by Theorem 3.4 and the fact that  $\text{Tr } F = 0$ . Then since the expressions  $C_0(\log A, A) - C_0(\log A, P)$ ,  $C_0(\log B, B) - C_0(\log B, P)$  and  $C_0(\log AB, AB) - C_0(\log AB, P)$  are locally determined by Corollary 3.3 (ii), we conclude the local determinedness of

$$\log \det AB - \log \det A - \log \det B = C_0(\log AB, AB) - C_0(\log A, A) - C_0(\log B, B).$$

When  $n$  is odd and  $A$  and  $B$  are even-even, then (cf. Corollary 3.6 (i))

$$\log \det AB - \log \det A - \log \det B = \text{TR}(\log AB - \log A - \log B) = 0,$$

since it is locally determined (local contributions give zero because of parity), so  $\det AB = \det A \det B$  then, as originally shown in [KV, Th. 7.1].

## References

- [CDMP] A. Cardona, C. Ducourtioux, J. P. Magnot and S. Paycha, *Weighted traces on algebras of pseudodifferential operators*, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **5** (2002), 503–540.
- [CDP] A. Cardona, C. Ducourtioux and S. Paycha, *From tracial anomalies to anomalies in quantum field theory*, *Comm. Math. Phys.* **242** (2003), 31–65.
- [CFNW] M. Cederwall, G. Ferretti, B. Nilsson, A. Westerberg, *Schwinger terms and cohomology of pseudodifferential operators*, *Comm. Math. Phys.* **175** (1996), 203–220.
- [G1] G. Grubb, *Functional Calculus of Pseudodifferential Boundary Problems*, *Progress in Math. vol. 65, Second Edition*, Birkhäuser, Boston, 1996, first edition issued 1986.
- [G2] ———, *Logarithmic terms in trace expansions of Atiyah-Patodi-Singer problems*, *Ann. Global Anal. Geom.* **24** (2003), 1–51.
- [G3] ———, *Spectral boundary conditions for generalizations of Laplace and Dirac operators*, *Comm. Math. Phys.* **240** (2003), 243–280.
- [G4] ———, *On the logarithm component in trace defect formulas*, *Comm. Part. Diff. Equ.* **30** (2005), 1671–1716.

- [G5] ———, *Remarks on nonlocal trace expansion coefficients*, arXiv: math.AP/0510041, to appear in a proceedings volume in honor of K. Wojciechowski, “Analysis and Geometry of Boundary Value Problems”, World Scientific.
- [GH] G. Grubb and L. Hansen, *Complex powers of resolvents of pseudodifferential operators*, Comm. Part. Diff. Eq. **27** (2002), 2333–2361.
- [GSc] G. Grubb and E. Schrohe, *Traces and quasi-traces on the Boutet de Monvel algebra*, in Ann. Inst. Fourier **54** (2004), 1641–1696.
- [GS1] G. Grubb and R. Seeley, *Weakly parametric pseudodifferential operators and Atiyah-Patodi-Singer boundary problems*, Invent. Math. **121** (1995), 481–529.
- [GS2] ———, *Zeta and eta functions for Atiyah-Patodi-Singer operators*, J. Geom. An. **6** (1996), 31–77.
- [Gu] V. Guillemin, *A new proof of Weyl’s formula on the asymptotic distribution of eigenvalues*, Adv. Math. **102** (1985), 184–201.
- [H] J. Hadamard, *Le Problème de Cauchy et les Équations aux Dérivées Partielles Linéaires Hyperboliques*, Hermann, Paris, 1932.
- [K] C. Kassel, *Le résidu non commutatif [d’après M. Wodzicki]*, Astérisque **177–178** (1989), 199–229; Séminaire Bourbaki, 41ème année, Exposé no. 41, 1988–99.
- [KV] M. Kontsevich and S. Vishik, *Geometry of determinants of elliptic operators*, Functional Analysis on the Eve of the 21’st Century (Rutgers Conference in honor of I. M. Gelfand 1993), Vol. I (S. Gindikin et al., eds.), Progr. Math. 131, Birkhäuser, Boston, 1995, pp. 173–197.
- [L] M. Lesch, *On the noncommutative residue for pseudodifferential operators with log-poly-homogeneous symbols*, Ann. Global Anal. Geom. **17** (1999), 151–187.
- [Lo] P. Loya, *The structure of the resolvent of elliptic pseudodifferential operators*, J. Funct. Anal. **184** (2001), 77–134.
- [MN] R. Melrose and V. Nistor, *Homology of pseudodifferential operators I. Manifolds with boundary*, manuscript, arXiv: funct-an/9606005.
- [M] J. Mickelsson, *Schwinger terms, gerbes and operator residues*, Symplectic Singularities and Geometry of Gauge Fields (Warsaw 1995), Banach Center Publications, vol. 39, 1997, pp. 345–361, arXiv: hep-th/9509002.
- [O1] K. Okikiolu, *The Campbell-Hausdorff theorem for elliptic operators and a related trace formula*, Duke Math. J. **79** (1995), 687–722.
- [O2] ———, *The multiplicative anomaly for determinants of elliptic operators*, Duke Math. J. **79** (1995), 723–750.
- [O3] ———, *Critical metrics for the determinant of the Laplacian in odd dimensions*, Annals of Math. **153** (2001), 471–531.
- [RS] D. Ray and I. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Adv. in Math. **7** (1971), 145–210.
- [Sc] E. Schrohe, *Complex powers of elliptic pseudodifferential operators*, Integral Eq. Oper. Th. **9** (1986), 337–354.
- [Sco] S. Scott, *Eta forms and the Chern character* (to appear).
- [S] R. T. Seeley, *Complex powers of an elliptic operator*, Amer. Math. Soc. Proc. Symp. Pure Math. **10** (1967), 288–307.
- [W] M. Wodzicki, *Local invariants of spectral asymmetry*, Invent. Math. **75** (1984), 143–178.

COPENHAGEN UNIV. MATH. DEPT., UNIVERSITETSPARKEN 5, DK-2100 COPENHAGEN, DENMARK.

*E-mail address:* grubb@math.ku.dk